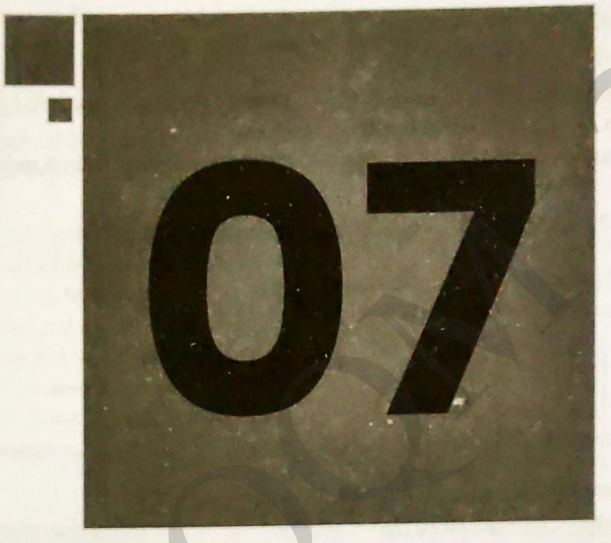

Combinatorics

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Chapter 07

Combinatorics

1. Combinatorics

Combinatorics is the field of mathematics dealing with the problems of selection, arrangements, and operations within a finite or discrete system. In this chapter, we will learn how to count the number of ways in which a particular job can be done. The concept is to break down the job into little sub jobs & then use **Fundamental Principles of Counting.**
First let's see the fundamental principles of counting & then pick arrangements & selections one by one & study their approaches & results.

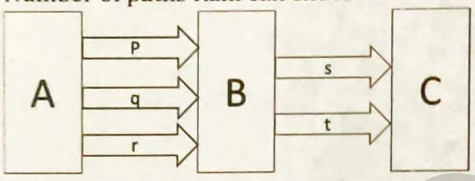
7.1 Fundamental Principles of Counting

(a) Fundamental Principle of Multiplication:

If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then the total number of ways in which both can occur in definite order are $\mathbf{m \times n}$.

Example 1: Ram has to go from place A to place C via place B. If number of paths from A to B are 3 & from B to C are 2, then find number of paths Ram can choose from A to C.

Sol. Event 1: Ram goes from A to B has 3 ways
Event 2: Ram goes from B to C has 2 ways
Thus, using fundamental principle of multiplication (FPM),
Number of paths Ram can choose $= 3 \times 2 = 6$



These 6 different paths are {p-s, p-t, q-s, q-t, r-s, r-t}.

Note: This Principle is also applicable for more than 2 events.

(b) Fundamental Principle of Addition:

If an event can occur in 'm' different ways and another event independent to the first can occur in 'n' different ways, then the total number of ways in which one of them can occur is $\mathbf{m + n}$.

Example 2: In a book shop, there are 3 different chemistry books and 2 different physics books. Find the number of ways to buy one book.

Sol. E_1: To buy chemistry book has 3 ways
E_2: To buy physics book has 2 ways
Thus, using fundamental principle of addition (FPA),
Number of ways to buy one book $= 3 + 2 = 5$.

Note: This Principle is also applicable for more than 2 events.

Factorial: Factorial of a natural number n is the product of natural numbers from 1 to n. It is represented by n!

$\underline{|n}$ or $n! = n \times (n-1) \times (n-2) \times \ldots\ldots 2 \times 1$

Eg: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120, 4! = 4 \times 3 \times 2 \times 1 = 24, 1! = 1$

Note: It's a convention to take $0! = 1$

Observation: $n! = n \times (n-1)!$

Example 3: How many flags containing 3 stripes of different colours can be made using 6 colours.

Sol. Since there are 6 different colours
Thus, first stripe can be taken in 6 ways
Second stripe can be taken in 5 ways
Third stripe can be taken in 4 ways

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Thus, using fundamental principle of multiplication (FPM),
Number of total flags $= 6 \times 5 \times 4 = 120$

Example 4: A room has 5 doors. In how many ways a man can enter the room through one door & come out of different door?

Sol. Since there are 5 doors, a man can enter from either one, but he needs to come out by a different door, therefore, to come out he has 4 doors. Thus, using fundamental principle of multiplication (FPM),
Total number of ways $= 5 \times 4 = 20$

Example 5: In an exam there are 3 true/false questions. In how many ways one can attempt the paper if:
(a) each question has to be attempted
(b) one can attempt or leave the question
(c) at least one question has to be attempted

Sol. (a) Here, each question has to be attempted and job is done when all the 3 questions are answered, also each question has two ways to answer. Thus, using fundamental principle of multiplication (FPM), Number of ways $= 2 \times 2 \times 2 = 8$

(b) Here either we can leave the question or can attempt the question. So, there are three ways of attempting each question. Thus, using fundamental principle of multiplication (FPM), Number of ways $= 3 \times 3 \times 3 = 27$

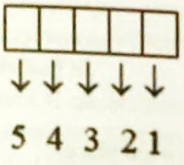
(c) Out of 27 ways obtained in previous part, there is only one way in which no question is attempted. So, answer of this part is $27 - 1 = 26$

Example 6: There are 5 questions, each having four options. In how many ways they can be attempted such that no two consecutive questions are marked with same options?

Sol. Here, first question can be done in 4 ways
Following it 2nd can be done in 3 ways
Following it 3rd can be done in 3 ways and so on
Remaining 4th and 5th can be done in 3 ways each
Total number of ways $= 4 \times 3 \times 3 \times 3 \times 3 = 324$

Example 7: How many 5 letter words can be formed using letters of 'KNIFE' if repetition of letters is not allowed?

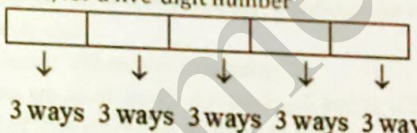
Sol: Here,



Thus, total number of words $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Example 8: How many 5-digit numbers can be formed using 1, 2 & 3?

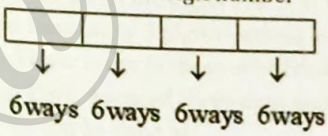
Sol. Here, digits to be used are 1, 2 and 3
Thus, for a five-digit number



Total numbers $= 3^5 = 243$

Example 9: How many 4-digit numbers with possible repetition of digits can be made with no digit less than 4.

Sol. Digits not less than 4 are 4, 5, 6, 7, 8, 9. i.e. 6 digits
Thus, for a four-digit number



Total numbers $= 6 \times 6 \times 6 \times 6 = 1296$

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Example 10: An n-digit number is a positive number with exactly n-digits. Nine hundred distinct n-digit numbers are to be formed using only the three digits 2, 5 & 7. The smallest value of 'n' for which this is possible, is

Sol. Distinct n-digit number which can be formed using digits 2, 5 and 7 are 3^n

We have to find 'n' so that

$3^n \geq 900$

$\Rightarrow 3^{n-2} \geq 100$

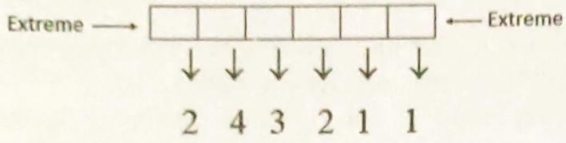
$\Rightarrow n-2 \geq 5$

$\Rightarrow n \geq 7$

So, the least value of n is 7

Example 11: How many 6 letter words can be formed jumbling the letters of 'JUNGLE' such that, vowels are at extreme positions.

Sol. Given 6 letters, out of which there are 2 vowels which can take extreme positions in 2 ways and remaining 4 can arrange as follows.



Total number of ways $= 2 \times 4 \times 3 \times 2 \times 1 \times 1 = 2 \times 4! = 48$

Example 12: How many words can be formed by rearranging letters of 'DELHI' such that vowels occupy odd places?

Sol. Method I

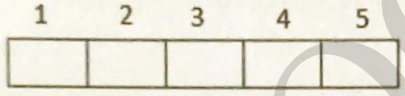
Here, first vowel has 3 options and following it 2nd vowel has two options

Then remaining 3 letters can be arranged in remaining 3 places in $3 \times 2 \times 1$ ways

Total number of ways $= 3 \times 2 \times 2 \times 1 = 6 \times 3!$

Method II

If 5 places can be represented by boxes as numbered below



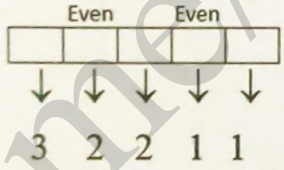
Then two vowels can take position as 1-3 or 3-5 or 1-5

And remaining places will be arranged by putting 3 remaining letters

Total number of ways $= 3 \times 2 \times 3 \times 2 \times 1 = 6 \times 3!$

Example 13: How many words can be formed rearranging letters of 'DELHI' such that vowels occupy even places?

Sol. There are two vowels (E, I) they can occupy even place as



And remaining 3 places will be taken by 3 letters in $3 \times 2 \times 1$ or 3! ways

Total number of words $= 2 \times 3!$

Example 14: How many ordered pairs (a, b) are there such that LCM of a & b is $2^3 \times 5^2$

Sol. For L.C.M to be $2^3 \times 5^2$

a and b must be of the form $a = 2^p5^q$, $b = 2^r5^s$

Possible powers of 2 in 'a' and 'b' are

(p, r) = (0, 3), (1, 3), (2, 3), (3, 3), (3, 0), (3, 1), (3, 2)

Thus, total 7 ways

Similarly for 5, we have

(q, s) = (0, 2), (1, 2), (2, 2), (2, 0), (2, 1)

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Thus, total 5 ways

Using Fundamental principle of multiplication (FPM)

Total number of ways = $5 \times 7 = 35$

Example 15: Five players P_1, P_2, P_3, P_4 and P_5 have to sit in a five-seater car. Only P_1, P_2, P_3 knows how to drive. In how many ways they can be arranged so that P_4 & P_5 sit in back row. It is given that front row has two seats & back row has three seats.

Sol. We want P_4 and P_5 to sit in back. Thus, P_5 has three options following it P_4 has two options.

And remaining P_1, P_2, P_3 can arrange in remaining 3 seats in $3 \times 2 \times 1$ ways

Total number of ways = $6 \times 3! = 36$

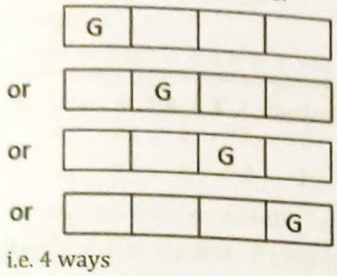
Example 16: How many four lettered words containing 'G' can be formed using letters of DAUGHTER

(a) Repetition not allowed

(b) Repetition allowed

Sol. (a) **Method I:**

Four possibilities are there:



i.e. 4 ways

And remaining 3 empty positions can be taken by remaining 7 letters in $7 \times 6 \times 5$ ways

Total number of words = $4 \times 7 \times 6 \times 5 = 840$

Method II:

Total number of four letter words formed by 8 letters = $8 \times 7 \times 6 \times 5 = 1680$

All four letters word formed by 7 letters (except G) = $7 \times 6 \times 5 \times 4 = 840$

Total words which have G = $1680 - 840 = 840$

Method III

Here, G can accommodate in any of the four places, thus G has 4 ways

Remaining 3 letters will be taken by remaining 7 letters in $7 \times 6 \times 5$ ways

Total number of ways = $4 \times 7 \times 6 \times 5 = 840$

(b) Total number of four letter words from given 8 letters = $8 \times 8 \times 8 \times 8 = 8^4$

Also, number of four letter words from 7 given letters (excluding G) = 7^4

Total number of words having G = $8^4 - 7^4 = 4096 - 2401 = 1695$

Example 17: How many 3 letter words can be made using letters of DAUGHTER such that if word contains 'G' then it must contain H also.

Sol. **Case I:** When word contains G

So, we need words containing G and H, G has 3 options and H has 2 options

Left one place can be filled in 6 ways

Total number of words containing G and H = $3 \times 2 \times 6 = 36$

Case II: When word does not contain G = $7 \times 6 \times 5 = 210$

Total number of 3 letter word = $210 + 36 = 46$

Example 18: Given the digits 1, 2, 3, 4, 5 & 9

(a) How many 3 digit numbers less than 600 can be formed if each digit is used at most once

(b) How many of these are divisible by 5

Sol. (a) Here, we need three digit number less than 600, using 1, 2, 3, 4, 5 and 9

This can be done as

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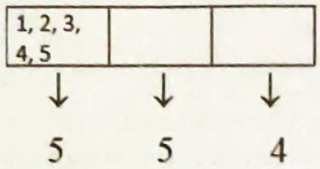


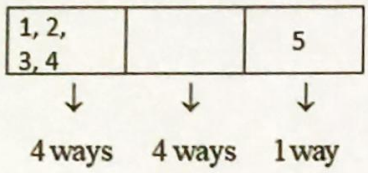
(as 9 can't come in hundreds place but can come in other places)

Total number of numbers = $5 \times 5 \times 4 = 100$

(b) For number divisible by 5

One's place should be taken by 5

This can be done as



Total numbers = $4 \times 4 = 16$

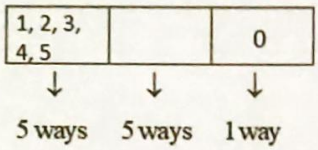
Example 19: How many natural numbers less than 600 & divisible by 5 can be formed using 0, 1, 2, 3, 4, 5, 9

(a) Repetition of digits not allowed

(b) Repetition of digits allowed

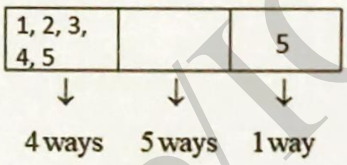
Sol (a) For a number to be divisible by 5, it should have 0 or 5 in its one's place

Case I:



$= 5 \times 5 \times 1 = 25$ numbers

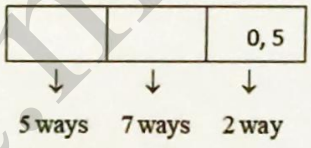
Case II:



$= 4 \times 5 \times 1 = 20$ numbers

Total number = 20 + 25 = 45

(b) For a number to be divisible by 5, it should have 0 or 5 in its one's place



$= 5 \times 7 \times 2 = 70$ numbers

Example 20: How many four-digit numbers can be formed using 1, 2, 3, 4, 5 which are divisible by 4, if repetition of digits is not allowed?

Sol. For a number to be divisible by 4

Last two digits should be divisible by 4

Here digits given are 1, 2, 3, 4, 5

Thus, last two digits can be 12, 24, 32, 52 i.e. 4 ways

And remaining two places can be filled by 3×2 ways

Total number = $4 \times 3 \times 2 = 24$

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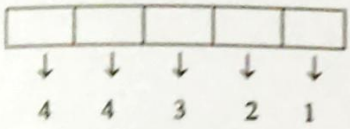
Example 21: How many four digit numbers can be formed using 1, 2, 3, 4, 5 which are divisible by 4, if repetition of digits is allowed?

Sol. For number divisible by 4 last two digits can be 12, 24, 32, 52, 44
i.e. 5 ways
And remaining two places can be taken by 5×5 ways
Total number of ways $= 5 \times 5^2 = 125$

Example 22: A five digit number divisible by 3 is to be formed using 0, 1, 2, 3, 4 & 5 without repetition. The total number of ways this can be done is

Sol. We know that a number is divisible by 3 if the sum of its digit is divisible by 3.
Now out of 0, 1, 2, 3, 4, 5 if we take 1, 2, 3, 4, 5 or 0, 1, 2, 4, 5 then 5-digit numbers will be divisible by 3
Case I
Total number of five-digit number formed using the digits 1, 2, 3, 4, 5
$= 5 \times 4 \times 3 \times 2 \times 1 = 120$
Case II:
Taking 0, 1, 2, 4, 5
Total number is $= 4 \times 4! = 96$

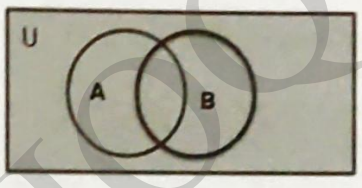


Thus, total number divisible by $3 = 120 + 96 = 216$

Inclusion-Exclusion Principle

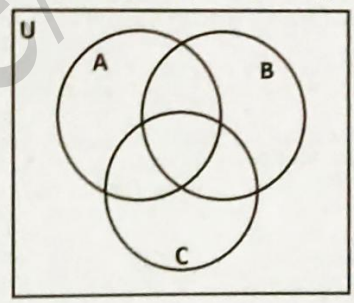
If in the statement of question, you see "Either or" or "neither nor" then always try using:

(a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$



(b) $n(A^c \cap B^c) = \text{Total} - n(A \cup B)$

(c) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - (n(A \cap B) + n(B \cap C) + n(A \cap C)) + n(A \cap B \cap C)$



In general
$n(A_1 \cup A_2 \cup A_3 \cup \ldots\ldots \cup A_n) = \sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1}\, n(A_1 \cap A_2 \cap A_3 \cap \ldots \cap \ldots \cap A_n)$

Example 23: How many four digit numbers can be formed which
(a) either begin with 3 or end with 8
(b) neither begin with 3 not end with 8

Sol. (a) Let n(A) be natural numbers that begin with 3 & let n(B) be natural numbers that end with 8

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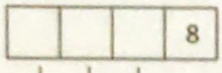
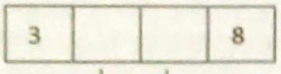
1 way 10 10 10

$\therefore n(A) = 10^3 = 1000$



9 10 10

$n(B) = 900$



10 10

$n(A \cap B) = 100$

Number either begin with 3 or with 8 is n $(A \cup B)$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 1000 + 900 - 100 = 1800$

(b) Number either begin with 3 nor and with 8 is $n(A^c \cap B^c)$

$n(A^c \cap B^c)$ = Total 4 digit number – $n(A \cup B)$

$= 9 \times 10^3 - 1800 = 7200$

Example 24: Four distinct dice are thrown simultaneously. Find number of different possible cases (with respect to the numbers coming on upper faces of the dice) in which 1 and 2 must appear.

Sol. $n(A)$ = number of cases with 1 missing = 5^4

$n(B)$ = number of cases with 2 missing = 5^4

$n(A \cap B)$ = number of cases with 1 and 2 missing = 4^4

$n(A \cup B) = 2 \times 5^4 - 4^4$

Required number of ways = $n(A' \cap B')$ = Total - $n(A \cup B)$

$= 6^4 - 2 \times 5^4 + 4^4$

Exercise–1

1. There are 6 roads between A & B and 4 roads between B & C.
   (i) In how many ways can one drive from A to C by way of B?
   (ii) In how many ways can one drive from A to C and back to A, passing through B on both trips?
   (iii) In how many ways can one drive the circular trip described in (ii) without using the same road more than once.
2. (i) How many car number plates can be made if each plate contains 2 different letters of english alphabet, followed by 3 different digits.
   (ii) Solve the problem, if the first digit cannot be 0.
3. (i) Find the number of four letter word that can be formed from the letters of the word HISTORY. (each letter to be used at most once)
   (ii) How many of them contain only consonants?
   (iii) How many of them begin & end in a consonant?
   (iv) How many of them begin with a vowel?
   (v) How many contain the letter Y?
   (vi) How many begin with T & end in a vowel?
   (vii) How many being with T & also contain S?
4. In how many ways can 5 letters be mailed if there are 3 different mailboxes available if each letter can be mailed in any mailbox?

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5. Every telephone number consists of 7 digits. How many telephone numbers are there which do not include any other digits but 2, 3, 5 & 7?
6. How many of the arrangements of the letter of the word "LOGARITHM" begin with a vowel and end with a consonant?
7. In a telephone system four different letters P, R, S, T and the four digits 3, 5, 7, 8 are used. Find the maximum number of "telephone numbers" the system can have if each consists of a letter followed by a four-digit number in which the digit may be repeated.
8. Find the number of 5 lettered palindromes which can be formed using the letters from the english alphabets.
9. Number of ways in which 7 different colours in a rainbow can be arranged if green is always in the middle.
10. A letter lock consists of three rings each marked with 10 different letters. Find the number of ways in which it is possible to make unsuccessful attempts to open the lock.
11. It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?
12. In how many ways can clean & clouded (overcast) days occur in a week assuming that an entire day is either clean or clouded.
13. How many 4 lettered words starting with 'V' can be formed using letters of VACUITY if (a) Repetition not allowed (b) Repetition allowed
14. A new flag is to be designed with six vertical strips using some or all of the colours yellow, green, blue and red. Then, the number of ways this can be done such that no two adjacent strips have the same colour is _____.
15. In how many ways five friends F_1, F_2, F_3, F_4 & F_5 have to be arranged in a 5 seater car having 2 seats in front (including driver seat) & 3 seats in the back, such that F_1 doesn't drive. Also, in how many ways it can be done such that if F_1 drives then F_2 has to sit next to him.
16. How many 10 digit numbers can be made with odd digits so that no two consecutive digits are same.
17. Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.
18. Find all possible three digits even numbers which can be formed with the condition that if 5 is one of the digit, then 7 is the next digit.
19. How many natural numbers are there from 1 to 1000 which have none of their digits repeated?
20. If repetitions are not permitted
    (i) How many 3-digit numbers can be formed from the six digits 2, 3, 5, 6, 7 & 9?
    (ii) How many of these are less than 400?
    (iii) How many are even?
    (iv) How many are odd?
    (v) How many are multiples of 5?
21. Find the number of six-digit numbers that can be formed from the digits 1, 2, 3, 4, 5, 6 & 7 so that digits do not repeat and the terminal digits are even.
22. How many numbers between 400 and 1000 (both exclusive) can be made with the digit 2, 3, 4, 5, 6, 0 if
    (a) Repetition of digits not allowed (b) Repetition of digits is allowed
23. Find the number of 5-digit numbers such that the sum of their digits is even.
24. Find the number of 9-digit numbers that can be formed by using the digits 1, 2, 3, 4 & 5.
25. There are 11 players P_1, P_2,..., P_{11} in a team. How many batting orders are possible if
    (a) Either P_1 is at first position or P_5 is at last position
    (b) Neither P_1 is at first position nor P_5 is at last position.
26. How many of the 900 three-digit numbers have at least one even digit?
27. Find the number of natural numbers from 1000 to 9999 (both inclusive) that do not have all 4 different digits.
28. Find number of natural numbers in the first 10^6 natural numbers which are neither a perfect square nor a perfect cube.
29. Number of numbers in the list {1, 2, 3,..., 500} which are either divisible by 2 or by 3 or by 7.

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7.2 Combinations

A Combination is the selection of some or all distinct objects from given set of distinct objects, irrespective of their arrangements or order in which they are selected. We are interested to find the number of different combinations(number of different selections).

Selection of Distinct Objects

(i) Number of ways to select r different items from n different items are ${}^{n}C_{r}$.

Here, ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$. For example, ${}^{10}C_{2} = \frac{10!}{2!8!}$

Note: ${}^{n}C_{r} = {}^{n}C_{n-r}$, ${}^{n}C_{0} = 1$, ${}^{n}C_{1} = n$

(ii) Total number of selections out of 'n' distinct objects includes the number of ways to select 0, 1, ..., n objects is equal to 2^n.

Number of non-empty selections are = $2^n - 1$

Proof:

Method I: Every element can be dealt in two ways. Either it will be selected or not. So, required ways are 2^n.

Method II: n(no selection) + n(1 object selected) + ... + n(all objects selected)

${}^{n}C_{0} + {}^{n}C_{1} + {}^{n}C_{2} + ...+ {}^{n}C_{n} = 2^n$

Example 1: Find the number of ways to select a team of 3 students out of 10 students.

Sol. ${}^{10}C_{3} = \frac{10!}{3!7!} = \frac{10.9.8}{6} = 120$

Example 2: A committee of 5 members is to be made out of 5 women & 6 men.

(a) In how many ways, committees can be made

(b) How many committees will contain 3 men and 2 women

(c) How many committees contain at most 2 women

(d) How many committees contains at least one woman

Sol. (a) Number of committee of 5 members is ${}^{11}C_{5} = 462$

(b) Clearly committees containing 3 men and 2 women = ${}^{6}C_{3} \times {}^{5}C_{2} = 200$

(c) Committees containing almost two women (i.e. 0 or 1 or 2 women)

$= {}^{6}C_{3} \times {}^{5}C_{0} + {}^{6}C_{4} \times {}^{5}C_{1} + {}^{6}C_{3} \times {}^{5}C_{2}$

$= 6 + 75 + 200$

$= 281$

(d) Committees containing at least one women =

Total number of committees – committees containing 0 women

$= 462 - {}^{6}C_{5}$

$= 462 - 6$

$= 456$

Example 3: A committee of 14 is to be formed from 9 women & 8 men. In how many ways this can be done if women should be in majority?

Sol. Committee of 14 can be formed where women are in majority as follows

Women(9)	Men (8)
9	5
8	6

Number of committees

$= {}^{9}C_{9} \times {}^{8}C_{5} + {}^{9}C_{8} \times {}^{8}C_{6}$

$= 56 + 252$

$= 308$

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Example 4: There are 6 happily married couples. How many different mixed double games can be played such that no couple plays in same game?

Sol. Here, 2 men can be selected in ${}^{6}C_2$ ways, following which 2 women out of 4 women (except wives of selected husband) can be selected in ${}^{4}C_2$ ways Among these 4 selected people 2 mixed double games can be played

Thus, total number of games $= {}^{6}C_2 \times {}^{4}C_2 \times 2 = 180$

Example 5: A student is allowed to select at most 'n' books from a collection of $2n + 1$ distinct books. If total number of ways in which he can select atleast one book is 63, then find value of n.

Sol. Number of ways in which a student can select at least one and atmost n books out of $2n + 1$ books is

$2^{n+1}C_1 + {}^{2n+1}C_2 + {}^{2n-1}C_3 + \ldots {}^{2n+1}C_n$

$= (1/2)[2 \times {}^{n+1}C_1 + 2 \times {}^{2n+1}C_2 + 2 \times {}^{2n+1}C_3 + \ldots + 2 \times {}^{2n+1}C_n]$

$= (1/2)[({}^{2n+1}C_1 + {}^{2n+1}C_{2n}) + ({}^{2n+1}C_2 + {}^{2n+1}C_{2n-1}) + ({}^{2n+1}C_3 + {}^{2n+1}C_{2n-2}) + \ldots + ({}^{2n+1}C_n + {}^{2n+1}C_{n+1})]$

$= (1/2)(2^{2n+1} - 1) = 2^{2n} - 1$

ATQ

$2^{2n} - 1 = 63$

$2^{2n} = 64 = 2^6$

$2n = 6$

$n = 3$

Example 6: In how many ways a team of 11 players can be selected out of 15 players such that:

(a) A particular player is never chosen

(b) Three particular players are always chosen

(c) Two particular players never come together in a team

(d) At least one of the 3 particular players is in the team

Sol. (a) Since a player is never chosen

Thus, selection will be taken from remaining 14 players

Number of ways $= {}^{15-1}C_{11} = {}^{14}C_{11} = 364$

(b) Here 3 players are always chosen

Thus, selection of only 8 players is left out of remaining 12 players

Number of ways $= {}^{15-3}C_{11-3} = {}^{12}C_8 = 495$

(c) Two particular players never come together such that we can have a team without them or taking any one of them

Thus, number of ways $= = {}^{15-2}C_{11} + {}^{2}C_1 \times {}^{13}C_{10}$

$= ({}^{13}C_{11}) + 2\,({}^{13}C_{10}) = 78 + 2\,(858) = 1794$

(d) Required number of ways

= Total number of teams – teams not having any of those three particular players

$= {}^{15}C_{11} - {}^{15-3}C_{11}$

$= {}^{15}C_{11} - {}^{12}C_{11} = 1365 - 12 = 1353$

Example 7: Ram studies in 11th class. Strength of his class is 50. Teacher takes 4 students at a time to zoo for picnic. No four same students are selected more than once. How many times Ram will go to zoo for picnic?

Sol. Here the number of times Ram (i.e. a student) goes to the zoo is equal to the number of ways students can be selected who will accompany Ram i.e. ${}^{50-1}C_3 = {}^{49}C_3$

Example 8: There are 3 distinct physics books & 4 distinct maths books. In how many ways

(a) Books can be selected

(b) At least one book can be selected

(c) Selection can be made such that at least one maths book is selected

Sol. (a) Given, three distinct physics books and four distinct maths books

Required number of ways $= 2^7 = 128$

(b) At least one book can be selected $= 2^7 - 1 = 127$

(c) For at least one maths book to be selected

$= (2^3)(2^4 - 1)$

$= (8)(15) = 120$

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Example 9: Consider a set A = {1, 2, 3, 4}

(a) Find the number of subsets.

(b) How many ordered pairs (P, Q) can be made where P & Q are subsets of A & $P \cap Q = \phi$?

(c) How many ordered pairs (P, Q) can be made where P & Q are subsets of A & $P \cap Q$ contains exactly one element?

Sol. (a) Given, A = {1, 2, 3, 4}

We need total number of selections. We have 2 options for each of the element. i.e. either to select it or not.

So, the number of subsets = $2^4 = 16$

(b) Here, every element (let's say x) can be dealt in following 3 ways so that $P \cap Q = \phi$

$x \notin P$ and $x \in Q$

$x \in P$ and $x \notin Q$

$x \notin P$ and $x \notin Q$

$\therefore$ Number of ways = 3^4

(c) The one common element can be selected in 4 ways

Remaining 3 elements each can be dealt in 3 ways each (as in previous part)

Required number of ways = 4×3^3

$= 4 \times 27$

$= 108$

Selection of Alike Items

(a) Number of ways to select r identical items from n identical items = 1

(b) Total number of selections (selecting 0, 1, ..., n items) of n identical items = n + 1. This includes no selection also.

Ex: Number of ways to select AA from A, A, A, A, A is 1. Total number of selections is 6 and non-empty selections are 5(A, AA, AAA, AAAA, AAAAA).

(c) Total number of selection of (a + b + c + d) items in which a are alike of one kind, b are alike of 2nd kind, c are alike of 3rd kind and d are distinct items = $(a + 1)(b + 1)(c + 1)2^d$

Number of non-empty selections = $(a + 1)(b + 1)(c + 1)2^d - 1$

Example 10: Find the number of ways of selecting at least one letter from A, A, A, B, B, C, C, C, C

Sol. $(3 + 1)(2 + 1)(4 + 1) - 1 = 59$

Example 11: Number of ways of selecting two letters out of A, A, A, B, B, B, C, C, C is _____

Sol. There are six ways AA, BB, CC, AB, BC, AC

Example 12: In how many ways we can select four letters out of 'ASSASSINATION'?

Sol. Here we have

$A \longrightarrow 3, S \longrightarrow 4, I \longrightarrow 2, N \longrightarrow 2, T \longrightarrow 1, O \longrightarrow 1$

Four letter selection can be done as

(i) All distinct = ${}^6C_4 = 15$

(ii) Two alike & two distinct = ${}^4C_1 \times {}^5C_2 = 40$

(iii) Two alike & two alike = ${}^4C_2 = 6$

(iv) Three alike & one distinct = ${}^2C_1 \times {}^5C_1 = 10$

(v) All alike = 1

$\therefore$ Total number of ways = 72

Geometrical Counting Problems Based on ${}^nC_r$

Example 13: There are 'n' points a circle

(a) Joining these points, how many lines will be made?

(b) Joining these points, how many triangles will be made?

(c) Joining these points, a polygon is made. Find the number of its diagonals.

(d) Find the maximum number of points of intersection of the lines (which formed by joining the points) inside the circle.

Sol. (a) Two points correspond to one line

Thus, from 'n' points on a circle,

Total number of lines = ${}^nC_2$

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(b) Three non-collinear points correspond to one triangle
Thus, from 'n' points on a circle,
Total number of triangles = ${}^{n}C_3$

(c) For a diagonal which is a line we require two points, but those selection will include sides of polygon also as they are all formed by selecting two points
Number of diagonals of 'n' sided polygon is ${}^{n}C_2 - n$.

(d) Four points on circle correspond to one point of intersection inside the circle, so the number of points of intersections are ${}^{n}C_4$.

Remark: Remember it as a result, that number of diagonals in n-sided polygon is ${}^{n}C_2 - n$.

Example 14: Consider 8 points, out of which 4 are collinear. Joining these points how many
(a) lines will be made
(b) triangles will be made

Sol. (a) **Method I:**
Given 4 points are collinear out of 8 points
Thus, lines formed by these 4 points will be same line
Thus, total number of different lines = ${}^{8}C_2 - {}^{4}C_2 + 1 = 22 + 1 = 23$
Method II:
Given 4 points are collinear out of 8 points
$\therefore$ Total number of lines = ${}^{4}C_2 + {}^{4}C_1 \times {}^{4}C_1 + 1 = 6 + 16 + 1 = 23$
(b) **Method I:**
Here also collinear points will not make any triangle
Thus, total number of triangles = ${}^{8}C_3 - {}^{4}C_3 = 56 - 4 = 52$
Method II:
Here we can take different selection from collinear points and non-collinear points,
Such that number of triangles will be = ${}^{4}C_1 \times {}^{4}C_2 \times {}^{4}C_2 \times {}^{4}C_1 + {}^{4}C_3$
$= 24 + 24 + 4 = 52$

Example 15: Find the number of Δ's that can be made joining marked points

Sol. There are a total 12 points out of which 4, 5 and 6 points are collinear on respective sides of the triangle.
Number of triangles = ${}^{12}C_3 - {}^{4}C_3 - {}^{5}C_3 - {}^{6}C_3$

Example 16: Find number of Δ's that can be made joining marked points.

Sol. There are in total 8 points out of which 4 are collinear in one line and 5 are collinear in one line
Number of triangles = ${}^{8}C_3 - {}^{4}C_3 - {}^{5}C_3$
$= 56 - 4 - 10 = 42$

Example 17: Consider 8 points, out of which 4 are collinear. Joining these points how many quadrilaterals will be made?

Sol. Here for quadrilateral we need 4 points
Thus, total number of quadrilaterals = ${}^{4}C_4 + {}^{4}C_4 \times {}^{4}C_1 + {}^{4}C_2 \times {}^{4}C_2 = 1 + 16 + 36 = 53$

Example 18: In a plane there are 8 lines, no two of which are parallel & no three are concurrent. How many triangles can be formed with their point of intersection as vertices?

Sol. When two lines intersect, we get one point of intersection
Total points when 8 lines intersect = ${}^{8}C_2 = 28$

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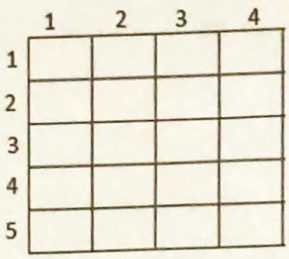
Every line will be intersected by 7 other lines, so on every line you will get ... points of intersection which are obviously collinear, being on a line

Number of triangles = ${}^{28}C_3 - 8({}^{7}C_3)$

Example 19: Given 6 horizontal & 5 vertical lines, how many rectangles & squares will be made by their intersection?

Sol. For rectangles, if we select any two horizontal line and any two vertical lines will get a rectangle

Total number of rectangles = ${}^{6}C_2 \times {}^{5}C_2$



For Squares:

1×1 squares $= 5 \times 4$ (choosing two consecutive horizontal and vertical lines)

2×2 squares $= 4 \times 3$ (choosing two H and V lines with the gap of one line)

3×3 squares $= 3 \times 2$ (choosing two H and V lines with the gap of two lines)

$4 \times 4 = 2 \times 1$ (choosing two H and V lines with the gap of three lines)

Total number of squares $= 5 \times 4 + 4 \times 3 + 3 \times 2 + 2 \times 1$

$= 20 + 12 + 6 + 2 = 40$

Exercise–2

1. A box contains two white balls, three black balls and four red balls. In how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw?
2. In how many ways can 5 colours be selected out of 8 different colours including red, blue and green,
   (a) If blue and green are always to be included
   (b) If red is to be always excluded
   (c) If red and blue are always included but green excluded?
3. A student has to answer 10 out of 13 questions in an examination. Find the number of ways in which he can answer if he must answer at least 3 of the first 5 questions.
4. A paper on mathematics consists of twelve questions divided into three parts A, B and C, each containing four questions. In how many ways can an examinee answer five questions, selecting at least one from each part?
5. A woman has 11 close friends. Find the number of ways in which she can invite 5 of them to dinner, if two particular friends out of them are not on speaking terms and will not attend together.
6. A rack has 5 different pairs of shoes. Find the number of ways in which 4 shoes can be chosen from it so that there will be no complete pair.
7. Everybody in a room shakes hands with everybody else. The total number of handshakes is 66. Find the total number of persons in the room.
8. Three ladies have each brought a child for admission to a school. The head of the school wishes to interview six people one by one, taking care that no child is interviewed before his/her mother. In how many different ways can the interviews be arranged?
9. There are 5 different physics, 3 different chemistry and 4 different math books. In how many ways books can be selected:
   (a) If at least one book is selected.
   (b) If at least one book is selected from each subject.
   (c) If at least one and at most two books are selected from each subject.
10. The interior angles of a regular polygon measure 150° each. Find the number of diagonals of the polygon.
11. 18 points are indicated on the perimeter of a triangle ABC (see figure). How many triangles are there with vertices at these points?

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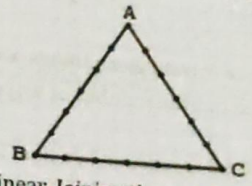
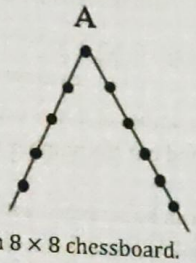


12. Consider 9 points out of which 4 are collinear. Joining these points how many
    (a) Lines will be made (b) Triangles will be made
    (c) Quadrilaterals will be made (d) Pentagons will be made
13. Triangles are formed joining the marked points as vertices, when
    (i) A is excluded &
    (ii) A is included
    Find ratio of number of triangles.



14. How many squares & rectangles are there in 8×8 chessboard.
15. There are n concurrent lines and another line parallel to one of them. The number of different triangles that will be formed by the $(n + 1)$ lines, is _____.
16. In how many ways we can pick 5 letters out of ARRANGEMENT.
17. Find the total number of selections of fruit which can be made from 3 bananas, 4 apples and 2 oranges.

7.3 Selection and Arrangement

Now, that we are completely equipped with ${}^{n}C_{r}$, let's do some examples using selection & arranging both together.

Selection and Arrangement of Distinct Objects

As if we try to observe number of 4 letter words that can be made using letters of DAUGHTER, repetition not allowed, can also be found as ${}^{8}C_{4} \times 4!$

Example 1: In how many ways we can select 3 boys & 2 girls out of 5 boys & 4 girls & arrange them on 5 seats?

Sol. Here 3 boys can be selected out of 5 boys in ${}^{5}C_{3}$ ways

2 girls out of 4 girls in ${}^{4}C_{2}$ ways

Now these selected 5 can be arrange on 5 seats in 5! ways

Total number of ways $= {}^{5}C_{3} \times {}^{4}C_{2} \times 5!$

$$= {}^{5}C_{3} \times {}^{4}C_{2} \times 5! = 10 \times 6 \times 120 = 7200$$

Example 2: In how many ways 3 couples can be arranged in a queue from left to right such that no husband is standing before his wife?

Sol. Select 2 places out of 6

First couple can be arranged there in one way. Following it select 2 places out of left 4 places.

Second couple can be arranged there in 1 way.

Following it third couple can be arranged at left 2 places in 1 way

$= {}^{6}C_{2} \times {}^{4}C_{2} \times 1 = 90$

Example 3: How many batting orders of 11 players can be made out of 15 players such that 2 particular players are always among the selected 11 players?

Sol Given two particular players are always selected

Thus, batting order of 11 players out of 15 can be done as

$$= {}^{15-2}C_{11-2} \times 11! = {}^{13}C_{9} \times 11!$$

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Gap Method & Block Method

Gap method: This method is used, when no two of certain given things occur together.

Let's say if there are m + n things, we want to find out the number of ways to arrange these in a row, so that no two of m things occur together.

(i) First of all, put the 'n' things on which there is no restriction in a line. These 'n' things can be arranged in n! ways.

(ii) Then count the number of gaps between every two of 'n' things on which there is no restriction including end positions. Number of such gaps will be (n + 1).

(iii) Number of ways to arrange the m things in these gaps $= {}^{n+1}C_m \times m!$

Required number of ways $= n! \times {}^{n+1}C_m \times m!$

Example 4: For example, if we have to arrange seven students A, B, C, D, E, F, G in row such that no two of A, C and E are together.

Sol. There are 4 students without restriction. So, we can arrange them in 4! ways.

Number of gaps between them = 5

Number of ways to arrange the A, C, E in these gaps $= {}^5C_3 \times 3!$

Therefore, number of ways $= 4! \times {}^5C_3 \times 3!$

Block Method: This method is used when certain number of things occur together.

Let's say if there are m + n things, and we want to find out the number of ways to arrange these in a row, so that m things occur together.

(i) First of all, consider a block of 'm' things. They can be arranged within themselves = m!

(ii) Now we have n + 1 things, 'n' things without restriction and one block. Number of ways to arrange them = (n + 1)!

(iii) Total Arrangements are = m!(n + 1)!

Example 5: For example, if we have to arrange seven students A, B, C, D, E, F, G in row such that A, C and E are together.

Sol. ACE get arranged in 3! ways. Now we have 4 students & a block of 3 students, i.e. 5 objects whose number of arrangements are = 5!. Total number of arrangements are = 3!5!

Note: n(Number of arrangements when not all m things together) = n(Number of arrangements without restriction) - (Number of arrangements when all m things are together)

Example 6: In how many ways 7 boys & 4 girls can be arranged in a row such that

(a) No two girls are together

(b) All girls are together

(c) Not all girls are together

Sol. (a) If four girls (G_i) are arranged between boys (B_i) then they will be never together if they sit in gaps among boys

So, arranging boys first in 7!, we get 8 gaps.

Select 4 gaps for girls in ${}^8C_4$ ways

Now arrange girls in these gaps by 4! ways

$B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times B_6 \times B_7$

$\therefore$ Total number of ways $= {}^8C_4 \times 4! \times 7!$

(b) Lets 'tie' all girls together and make them as one unit

$G_1, G_2, G_3, G_4, B_1, B_2, B_3, B_4, B_5, B_6, B_7$

Thus, now we have total 8 things to arrange and girls can mutually arrange among themselves.

Total number of ways $= 8! \times 4!$

(c) Not all girls together = Total arrangement – All girls together = 11! – 8!4!

Example 7: There are 2 lions, 3 elephants and 4 horses in a circus. In how many ways they can be arranged in a line so that

(a) No two lions are together

(b) Animals of same category are together

Sol. (a) For no two lions together

$\times E_1 \times E_2 \times E_3 \times H_1 \times H_2 \times H_3 \times H_4 \times$

Lions can take any of the 8 places in ${}^8C_2$ ways

Then all can mutually arrange in 7!2!

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Total number of ways $= {}^{8}C_2 \times 7! \times 2!$

(b) Animals of same category are together

L_1, L_2 E_1, E_2, E_3 H_1, H_2, H_3, H_4

Thus, number of arrangements is $= 3!(2!3!4!)$

$= 6(2 \times 6 \times 24)$

$= 1728$

Example 8: There are 4 boys and 4 girls. In how many ways they can be arranged such that

(a) No two girls are together

(b) Girls & boys are alternate

Sol: (a) Using Gap method

$\times B_1 \times B_2 \times B_3 \times B_4 \times$

If girls are placed in these gaps, then no two girls will be together

Thus, total number of ways $= {}^{5}C_4 \times 4!4! = 5 \times (4!)^2$

(b) Girls and boys are alternate in either of two ways as

$B_1G_1B_2G_2B_3G_3B_4G_4$

or $G_1B_1G_2B_2G_3B_3G_4B_4$

Total number of ways $= 2 \times (4!)^2$

Example 9: In how many ways 3 girls & 4 boys can be arranged such that girls and boys are alternate?

Sol. Here there is only one way in which 3 girls and 4 boys can be arranged

i.e. $B_1G_1B_2G_2B_3G_3B_4$

Total number of ways $= 4!3!$

Arrangements If Few Objects are Alike

Number of ways of arranging 'p' elements in a row if out of 'p' elements 'm' are alike, 'n' are alike & rest are distinct, is given by $\frac{p!}{m! \times n!}$.

Proof: Lets first arrange 'm' alike elements on 'p' places $= {}^{p}C_m \times 1$

Now arrange 'n' alike elements on 'p – m' places $= {}^{p-m}C_n \times 1$

Now arrange the remaining elements on the remaining places $= (p - m - n)!$

Number of ways arranging all elements $= {}^{p}C_m \times 1 \times {}^{p-m}C_n \times 1 \times (p - m - n)! = \frac{p!}{m! \times n!}$

For example:

(a) MUMMY $\rightarrow \frac{5!}{3!}$ ways

(b) ARRANGE $\rightarrow \frac{7!}{2!2!}$ ways

(c) AABBB $\rightarrow \frac{5!}{2! \times 3!}$ ways

(d) AABBBCC $\rightarrow \frac{7!}{2!3!2!}$ ways

Example 10: In how many ways letters of ARRANGE can be arranged such that:

(a) Terminal positions are taken by A

(b) Vowels are always together

(c) Both A are together & both R are together

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Sol. (a) Here terminal positions are taken by 'A'

R, R, N, G, E

A		A

Number of arrangements $= \frac{5!}{2!} = 60$

(b) Here required condition is

R, R, N, G, A, A, E

Number of arrangements $= \frac{5!}{2!} \times \frac{3!}{2!} = 180$

(c) Here required condition is AA, RR, N, G, E

Number of arrangements $= 5! \times \frac{2!}{2!} \times \frac{2!}{2!} = 120$

Example 11: How many four letter words can be made out of 'ASSASSINATION'?

Sol. Here we have, A ⟶ 3, S ⟶ 4, I ⟶ 2, N ⟶ 2, T ⟶ 1, O ⟶ 1

Four letter word can be made as

(i) All distinct $= {}^6C_4 \times 4! = 360$

(ii) Two alike & two distinct $= {}^4C_1 \times {}^5C_2 \times \frac{4!}{2!} = 480$

(iii) Two alike & two alike $= {}^4C_1 \times \frac{4!}{2!2!} = 36$

(iv) Three alike & one distinct $= {}^2C_1 \times {}^5C_2 \times \frac{4!}{3!} = 40$

(v) All alike $= = 1 \times \frac{4!}{4!} = 1$

Total number of words = 917

Example 12: In how many ways letters of ASSASSINATION can be arranged such that no two S's are together?

Sol. Using gap method we can do arrangement as

× A × A × I × N × A × T × I × O × N ×

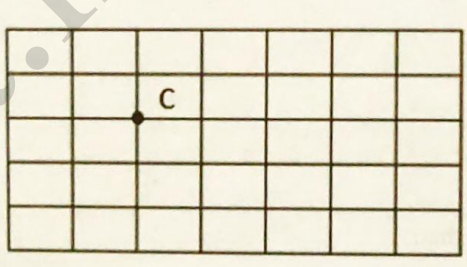
i.e. If we place S in any of these gaps, then they will be never together

Number of arrangements $= {}^{10}C_4 \times \frac{9!}{3!2!2!}$

Example 13: Find number of shortest paths

(a) from A to B

(b) from A to B via C



Sol. (a) There are 7 horizontal steps and 5 vertical steps

Any arrangement of 7H and 5V will give you a path from A to B.

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Total number of shortest paths are $= \frac{(7+5)!}{(7)!(5)!} = \frac{12!}{7!5!}$

(b) There are 2 horizontal steps and 3 vertical steps from A to C

Any arrangement of 2H and 3V will give you a path from A to C.

Similarly, there are 5 horizontal and 2 vertical steps from C to B.

Any arrangement of 5H and 2V will give you path from C to B.

Total number of shortest paths from A to B via C are

$$= \frac{(2+3)!}{(2)!(3)!} \times \frac{(5+2)!}{(5)!(2)!}$$

$$= \frac{5!}{2!3!} \times \frac{7!}{5!2!} = \frac{7!}{2!3!2!}$$

Dictionary Ranks

Example 14: (a) Find rank of SURAT in dictionary made up of all the words 5 letter words that can be formed using S, U, R, A, T; each letter used only once

(b) Letters of SURITI are permuted & all permutations are arranged as in an English dictionary. Find rank of SURITI

Sol. (a) The letters of word SURAT in alphabetical order are

A, R, S, T, U

Thus, words will be as follows

A					$\longrightarrow 4! = 24$
R					$\longrightarrow 4! = 24$
S	A				$\longrightarrow 3! = 6$
S	R				$\longrightarrow 3! = 6$
S	T				$\longrightarrow 3! = 6$
S	U	A			$\longrightarrow 2! = 2$
S	U	R	A	T	

Thus, before word SURAT came

We will have $= 4! + 4! + 3(3!) + 2!$

$= 24 + 24 + 18 + 2 = 68$

Thus, rank of SURAT is 69.

(b) The letters of word SURITI in alphabetical order are

I, I, R, S, T, U

Thus, words will be as follows:

I						$\longrightarrow 5! = 120$
R						$\longrightarrow \frac{5!}{2!} = 60$
S	I					$\longrightarrow 4! = 24$

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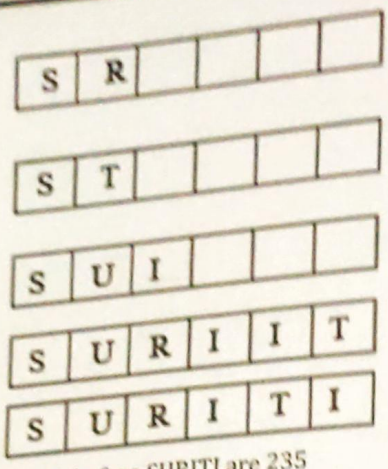

$\frac{4!}{2!} = 12$

$\frac{4!}{2!} = 12$

$3! = 6$

$1! = 1$

Words before SURITI are 235

Rank of SURITI is 236

Divisors

Let $N = P_1^{n_1} \times P_1^{n_2} \times \ldots\ldots \times P_m^{n_m}$

(i) Number of Divisors $= (n_1 + 1)(n_2 + 1)\ldots\ldots(n_m + 1)$

(ii) Sum of Divisors $= \left(\frac{P_1^{n_1+1} - 1}{P_1 - 1}\right) \times \left(\frac{P_2^{n_2+1} - 1}{P_1 - 1}\right) \times \ldots\ldots \times \left(\frac{P_m^{n_m+1} - 1}{P_m - 1}\right)$

(iii) Number of ways in which N can be resolved as a product of two factors is

$$\begin{cases} \frac{1}{2}(n_1+1)(n_2+1)\ldots\ldots(n_m+1); \text{ if N is not a perfect square} \\ \frac{1}{2}[(n_1+1)(n_2+1)\ldots\ldots(n_m+1)+1]; \text{ if N is a perfect square} \end{cases}$$

(iv) Number of ways in which N can be resolved as a product of two coprime factors is $= 2^{m-1}$

(v) Sum of all the divisors of $a^m \times b^n \times c^q$ is

$(a^0 + a^1 + a^2 + \ldots + a^m) \times (b^0 + b^1 + b^2 \ldots + b^n) \times (c^0 + c^1 + c^2 + \ldots + c^q)$

Example 15: For the number $2^4 \times 3^2 \times 5$, find

(a) Number of divisors

(b) Number of proper divisors

(c) Number of even divisors

(d) Number of divisors which are multiple of 6

Sol. (a) Given, number $2^4 \times 3^2 \times 5^3$

Number of divisors $= (4 + 1)(2 + 1)(3 + 1) = 5 \times 3 \times 4 = 60$

(b) Number of proper divisors $= (4 + 1)(2 + 1)(3 + 1) - 1 = 60 - 1 = 59$

(c) For even divisors, we should take at least one 2

Number of even divisors $= (4)(2 + 1)(3 + 1) = 4 \times 3 \times 4 = 48$

(d) Number of divisors which are multiple of 6, should contain at least one 2 and one 3

Number of divisors $= (4)(2)(3 + 1) = 32$

Example 16: Find the sum of all the divisors of $2^4 \times 3^2 \times 5^3$

Sol. Given, number is $2^4 \times 3^2 \times 5^3$

Sum of all the divisors

$$= 1\left(\frac{2^5 - 1}{2 - 1}\right) \times \left(\frac{3^3 - 1}{3 - 1}\right) \times \left(\frac{5^4 - 1}{5 - 1}\right)$$

$= (31)(12)(156)$

$= 62868$

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Exercise–3

1. Find the number of ways in which 4 boys and 2 girls (all are of different heights) can be arranged in a line so that boys as well as girls among themselves are in decreasing order of height (from left to right).
2. (a) Out of seven consonants and four vowels, the number of words of six letters, formed by taking four consonants and two vowels is_____ (Assume that each ordered group of letter is a word)
   (b) Find the number of permutations of the word "AUROBIND" in which vowels appear in an alphabetical order.
   (c) Find the number of permutations of the word "AUROBIND" in which no two vowels come together.
3. Find the number of ways in which letters of the word VALEDICTORY be arranged so that the vowels may never be separated.
4. Four visitors A, B, C & D arrive at a town which has 5 hotels. In how many ways can they disperse themselves among 5 hotels, if 4 hotels are used to accommodate them.
5. Number of six-digit numbers which have 3 digits even and 3 digits odd, if each is to be used at most once is _____.
6. If m denotes the number of 5 digit numbers if each successive digits are in their descending order of magnitude and n is the corresponding figure, when the digits are in their ascending order of magnitude then (m – n) has the value _____.
7. (a) If the letters of the word "VARUN" are written in all possible ways and then are arranged as in a dictionary, find the rank of the word VARUN.
   (b) The letters of the word RANDOM are written in all possible orders and these words are written out as in a dictionary, find the rank of the word RANDOM.
8. The total number of arrangements which can be made out of the letters of the word 'ALGEBRA', without altering the relative position of vowels and consonants is ____.
9. The total number of ways of arranging the letters AAAA BBB CC D E F in a row such that letters C are separated from one another is _____.
10. The number 2006 is made up of exactly two zeros and two other digits whose sum is 8. The number of 4-digit numbers with these properties (including 2006) is _____.
11. For the number $3^4 \times 5^2 \times 7^3$, find
    (a) Number of divisors
    (b) Number of even divisors
    (c) Number of divisors which are multiple of 21
    (d) Sum of all the divisors
12. (a) In how many ways the number 7056 can be resolved as a product of 2 factors?
    (b) Find the number of ways in which the number 300300 can be split into 2 factors which are relatively prime.
13. The number of odd proper divisors of $3^p \cdot 6^m \cdot 21^n$ is _____.

7.4 Distribution of Distinct Objects

Number of ways to distributing 'n' distinct objects among 'm' persons is m^n.

Proof: For each object, there are m options to distribute. Therefore, number of ways to distributing 'n' distinct objects among 'm' persons is m^n.

Remark: While distributing the distinct objects, always see the number of options that a given object to be distributed has.

Example 1: (a) In how many ways 5 distinct chocolates can be distributed among 4 students?
(b) In how many ways 3 distinct chocolates can be distributed among 4 students?
(c) If how many ways 3 distinct chocolates can be distributed among 4 students such that no student gets more than one chocolate?

Sol. (a) Here each chocolate can be distributed to 4 students in four ways

Total number of ways = 4^5

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(b) Here each chocolate can be distributed to 4 students in four ways

Total number of ways $= 4^3$

(c) Here, 3 students will get one chocolate each

Thus, total number of ways $= {}^4C_3 \times 3! = 24$

Distribution of Objects with Constraints (Formation of Groups)

Number of ways of distributing m + n distinct items into two groups, one contains m items and other contains n items is

$\frac{(m+n)!}{m!n!}$, here $m \neq n$.

Proof: Number of ways to select m items for one group $= {}^{m+n}C_m$

Remaining n items selected for the 2nd group = 1 way

$\Rightarrow$ Number of Distributions $= \frac{(m+n)!}{m!n!}$

Note: If m = n, then number of ways $= \frac{(2n)!}{2!n!n!}$. Here, we divide the expression by 2!, because the groups are identical, so each distribution is counted 2! times.

Above result can be extended to any number of groups, Let's try to understand this with the help of some examples given below:

$15 \rightarrow \{10,5\} \rightarrow \frac{15!}{5!10!}$

$15 \rightarrow \{7,3,5\} \rightarrow \frac{15!}{7!3!5!}$

$15 \rightarrow \{7,4,4\} \rightarrow \frac{15!}{7!4!4!2!}$

$15 \rightarrow \{5,5,5\} \rightarrow \frac{15!}{5!5!5!3!}$

$15 \rightarrow \{4,4,1,2,2,2\} \rightarrow \frac{15!}{4!4!1!2!2!2!3!2!}$

Note: If the groups further allotted to distinct persons or objects, then the number of ways of doing so = Number of ways to make groups × Arrangement of group among persons or objects.

Example 2: 6 distinct books have to be distributed among Ram, Shyam and Gopal. In how many ways it can be done such that

(a) Ram gets 1 book, Shyam gets 2 books & Gopal gets 3 books.

(b) One gets 3 books, one gets 2 books & another one gets 1 book.

(c) Ram gets 4 books while Shyam & Gopal gets 1 book each.

Sol. (a) Number of ways to form groups of 1 book, 2 books and 3 books $= \frac{6!}{1!2!3!}$

Number of arrangements of these groups among Ram, Sham and Gopal are = 1

$\Rightarrow$ Ram gets 1 book, shyam gets 2 books and Gopal gets 3 books is $= \frac{6!}{1!2!3!}$

(b) Here, anyone gets 3 books, one gets 2 books and another gets 1 book in

$= \frac{6!}{3!2!1!} \times 3! = \frac{6!}{2!}$

(c) Here, Gopal and Shyam get 1 book each, thus this can be done as

$= \frac{6!}{4!1!12!} \times 2! = \frac{6!}{4!}$

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Example 3: In how many ways a deck of 52 cards can be divided equally among 4 persons?

Sol We can divide 52 cards equally into four groups in $= \frac{52!}{(13!)^4 4!}$

Now, these four groups can be distributed among four players in 4! ways

Total number of ways of dividing the cards among four players equally is

$= \frac{52!}{(13!)^4 4!} \times 4! = \frac{52!}{(13!)^4}$

Example 4: In how many ways 8 passengers can be divided in 3 carriages such that:

(a) Each carriage contains at least two passengers.

(b) no carriage is empty.

Sol: (a)Since each carriage contains at least two passengers, then possible cases are (2, 2, 4) or (2, 3, 3)

Number of ways $= 3!\left[\frac{8!}{2!2!4!2!} + \frac{8!}{2!3!3!2!}\right]$

(b) **Method I:**

No carriage is empty means every carriage has at least one passenger.

This can be done as

(1, 3, 4), (2, 3, 3), (2, 2, 4), (5, 2, 1), (6, 1, 1) or (4, 3, 1)

Number of ways $= 3!\left[\frac{8!}{1!3!4!} + \frac{8!}{2!3!3!2!} + \frac{8!}{5!2!1!} + \frac{8!}{6!1!1!2!} + \frac{8}{4!3!2!}\right]$

Method II:

Number of ways = Total number of ways − (ways when one carriage is always empty)

− (ways when 2 carriages is always empty)

$= 3^8 - 3(2^8 - 2) - 3(1)$

Exercise–4

1. Find the number of ways in which n different prizes can be distributed amongst m (< n) persons if each is entitled to receive at most n – 1 prizes.
2. (a) Find the number of ways in which 12 balls can be divided between two friends, one receiving 8 and the other 4.
   (b) Find the number of ways in which 52 cards can be divided into 4 sets, three of them having 17 cards each and the fourth one having just one card.
3. Seven different coins are to be divided amongst three persons. If no two of the persons receive the same number of coins but each receives at least one coin and none is left over, find the number of ways in which the division may be made.
4. Number of different ways in which 8 different books can be distributed among 3 students, if each student receive at least 2 books is _____.
5. In a twelve-storey building, 10 people enter a lift cabin. It is known that they will leave the lift in groups of 3, 3 and 4 at different residential storeys. Find the number of ways in which they can get down.
6. Show that $(10!)^{9!}$ is a divisor of (10!)!

7.5 Distribution of Identical objects

Let's try to understand this with the help of an example.

Ex: If we have to find the number of ways to distribute 3 identical coins among three beggars. Let's have a sequence of letters cccpp, what are the different arrangements of these letters.

cccpp, ccppc, cppcc, ppccc, ccpcp, cpcpc, pcpcc, cpccp, pccpc, pcccp

if we count the number of c's before 1st p, number of c's between p's and number of c's after p's, we get (3, 0, 0), (2, 0, 1), (1, 0, 2), (0, 0, 3), (2, 1, 0), (1, 1, 1), (0, 1, 2), (1, 2, 0), (0, 2, 1), (0, 3, 0). But these order pairs are the different ways to

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distribute 3 identical coins among the three beggars. So, we can say that, number of ways to distribute 3 identical coins among three beggars = Number of ways to arrange 3c and 2p = $^5C_3$ or $^5C_2$

Result: Number of ways of distributing 'n' identical objects among 'r' persons, if each person may receive any number of objects is equal to arranging n c's and r – 1 p's = $^{n+r-1}C_{r-1}$

Ex: 10 identical one-rupee coins can be distributed among 3 beggars in $^{12}C_2$ ways

Remark: This result is majorly applied in finding non-negative integral solutions of equations.

The number of solutions of $x_1 + x_2 + \ldots + x_n = m$, here $x_1, x_2, \ldots, x_n$ are non negative integers are $^{n+m-1}C_m$

Example 5: Find the number of integral solutions of:

(a) $x+y+z=10$; $x,y,z \geq 0$

(b) $x+y+z=10$; $x \geq 1, y \geq 0, z \geq 0$

(c) $x+y+z=10$; $x \geq -2, y \geq 1, z \geq 0$

(d) $x+y+z=10$; $1 \leq x \leq 3$; $y \geq 0, z \geq 0$

(e) $2x+y+z=6$; $x,y,z \in W$

Sol. (a) Given $x+y+z=10$; $x,y,z \geq 0$

Number of non-negative integral solutions = $^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

(b) Given $x+y+z=10$; $x \geq 1, y \geq 0, z \geq 0$

Let $x-1=t; t \geq 0$

$\therefore 1+t+y+z=10$

$t+y+z=9; t \geq 0, y \geq 0, z \geq 0$

Number of non-negative integral solutions = $^{9+3-1}C_{3-1} = {}^{11}C_2 = 55$

(c) Given $x+y+z=10$; $x \geq -2, y \geq 1, z \geq 0$

Let $x+2=t$; $y-1=s$; $z-1=v$

$\therefore -2+t+1+s+1+v=10$

$t+s+v=10$; $t \geq 0, s \geq 0, v \geq 0$

Number of non-negative integral solutions = $^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

(d) Given $x+y+z=10$; $1 \leq x \leq 3$; $y \geq 0, z \geq 0$

x can take values 1, 2, or 3

Case I: $x = 1$,

$1+y+z=10$; $y \geq 0, z \geq 0$

$y+z=9$

Number of integral solutions = $^{9+2-1}C_{2-1} = {}^{10}C_1 = 10$

Case II: $x = 2$

$2+y+z=10; y \geq 0, z \geq 0$

$y+z=8$

Number of integral solutions = 9

Case III: $x = 3$

$3+y+z=10$; $y \geq 0, z \geq 0$

$y+z=7$

Number of integral solutions = 8

Total number of solutions = 10 + 9 + 8 = 27

(e) Given $2x+y+z=6$; $x,y,z \in W$

$\Rightarrow y+z=6-2x$

For $x = 0$, we have

$y+z=6-0=6$

Number of solutions = 7

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For $x = 1$, we have

$y + z = 6 - 2 = 6$

Number of solutions = 5

For $x = 2$, we have

$y + z = 6 - 4 = 2$

Number of solutions = 3

For $x = 3$, we have

$y + z = 6 - 6 = 0$

Number of solutions = 1

Total number of solutions = $7 + 5 + 3 + 1 = 16$

Example 6: Find the number of integral solutions of:

$x + y + z = 10; \quad x \geq 2, y \geq 1, z \geq 0$

Sol. Given $x + y + z = 10$; $x \geq 2, y \geq 1, z \geq 0$

Let $x - 2 = t, y - 1 = v \; t \geq 0, v \geq 0$

$\therefore 2 + t + 1 + v + z = 10$

$t + v + z = 7; t \geq 0, v \geq 0, z \geq 0$

$\therefore$ Number of non-negative integral solutions = ${}^{7+3-1}C_{3-1} = {}^{9}C_{2}$

Example 7: Suppose there are piles of red, blue & green balls, each containing at least 8 balls. In how many ways 8 balls can be selected (assuming balls of same colour to be alike)?

Sol. Let R represent red balls

B represent blue balls

G represent green balls

We require $R + B + G = 8$

Here $R \geq 0, B \geq 0, G \geq 0$

Number of ways = ${}^{8+3-1}C_{3-} = {}^{10}C_{2} = \frac{10!}{2!8!} = 45$

Example 8: Find total number of integral solutions for x, y & z such that $xyz = 24$

Sol. Given, $24 = 2^3 \times 3$

Now, consider x, y, z

Here, 3 can be given to any of three

Also, 2, 2, 2 can be distributed to x, y and z in $= {}^{3+3-1}C_{3-1} = {}^{5}C_{2}$ ways

Hence, total number of distributions for positive integers $= 3 \times {}^{5}C_{2} = 30$

Also, here any two of the three integers can be negative

Total number of distributions of integer $= 3 \times {}^{5}C_{2} \times {}^{3}C_{2} = 90$

Thus, total integral solution = 30 + 90 = 120

Exercise–5

1. The number of ordered triplets of positive integers which are solutions of the equation $x + y + z = 100$ is ____.
2. The total number of ways of selecting six coins out of 20 one rupee coins, 10 fifty paise coins and 7 twenty five paise coins is _____.
3. Find number of integral solutions of:
   (a) $x + y + z = 10; x \geq 0, y \geq 2, z > 0$
   (b) $x + y + z + w = 12; x \geq 4, y \geq -2, z \geq -3$
   (c) $2x + y + z = 10; x \geq 2, y \geq 10, z \geq 0$
   (d) $x + y + z = 12; 1 \leq x \leq 3, y \geq 0, z \geq 0$

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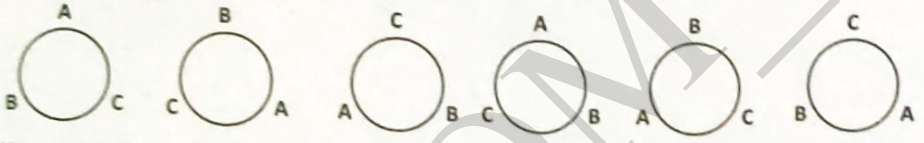
4. In how many ways 5 fruits can be selected out of unlimited identical bananas, identical apples & identical oranges.
5. Find number of non-negative integral solutions of $x+y+z \le 5$
6. Find the number of positive integer solutions of $x+y+z=10$, where x, y, z are unequal.
7. There are three piles of identical red, blue and green balls with each pile containing at least 10 balls. The number of ways of selecting 10 balls if twice as many red balls as green balls are to be selected is _____.
8. The number of ways in which 5 identical balls can be kept in 10 identical boxes, if not more than one can go into a box, is _____.
9. 5 balls are to be placed in 3 boxes. Each box can hold all the 5 balls. Number of ways in which the balls can be placed so that no box remains empty, if:
   (a) balls are identical, but boxes are different
   (b) balls are different, but boxes are identical
   (c) balls as well as boxes are identical
   (d) balls as well as boxes are identical, but boxes are kept in a row
10. Find the total number of ways in which n^2 number of identical balls can be put in n numbered boxed (1, 2, 3, ..., n) such that i^{th} box contains at least i number of balls.

7.6 Circular Arrangements

Let's understand this with an example, suppose we have 3 students A, B and C, we want to arrange them around a circular table. If we apply 3!, we get the following arrangements.



We can see first three arrangements are same as the order of letters are same. Similarly last three arrangements are also same with same reason. So, in actual we have only two arrangements. To avoid these extra counting, we first fix one student as reference and then arrange the others with respect to that reference.

Result: Number of ways of arranging n distinct objects in a circle is $(n-1)!$.

Proof: First fix one object as reference, now we have $n-1$ object to arrange with respect to reference, that can be done in $(n-1)!$ ways.

Remark: If clockwise & anticlockwise arrangements are considered (as in necklace and Garland etc) than number of arrangements are $\frac{(n-1)!}{2}$.

Example 1: In how many ways 5 distinct objects can be arranged in a
(a) circle (b) necklace

Sol. (a) Number of ways of arranging 5 distinct objects in circle $=(5-1)!=4!=24$

(b) Number of ways of arranging 5 distinct objects in necklace $=\frac{(5-1)!}{2}=12$

Example 2: Out of 10 beads of different colour, how many necklaces can be made, containing 4 beads of different colours?

Sol. Out of 10 beads of different colour, necklace containing 4 beads are

$={}^{10}C_4\frac{(4-1)!}{2}=630$

Example 3: In how many ways 6 boys & 4 girls can be arranged on circular table such that
(a) all girls are together
(b) no two girls are together

Sol. (a) Keeping all 4 girls together they will act as single person
Thus, number of arrangements on circular table is $=(7-1)!(4!)=6!\times 4!$

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(b) Here, there are 6 places between boys where if girls are arranged then no girls will be together

Number of arrangements $= {}^6C_4 \times 4! \times (6-1)! = \frac{6! \times 5!}{2}$

Example 4: Ram has called 10 of his friends including Shyam and Gopal at party. In how many ways Ram & his friends can sit on a round table such that Shyam and Gopal sit adjacent to Ram, one on each side?

Sol. Here, if we first fix Shyam and Gopal on either side of Ram as
S, R, G or G, R, S i.e. 2 ways
Such that they will be taken together as one and remaining 8 others are there
All 9 will be arranged on round table in $= (9-1)! \times 2 = 2 \times 8!$

Exercise–6

1. There are 16 men sitting on a round table. Find the number of ways to select 7 men such that no two are consecutive.
2. How many garlands of 6 flowers of different colours can be made out of 10 flowers of different colors?
3. In how many ways a team of 11 cricket players may sit on a round table so that captain, vice-captain & wicket keeper are together?
4. Find the number of ways in which seven persons can be arranged at a round table if two particular persons may not sit together.

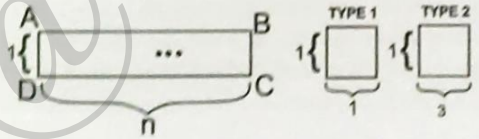
7.7 Recurrence Relation

A recurrence relation is an equation that defines a sequence based on a rule that gives the next term as a function of the previous term(s). For example, Fibonacci Sequence: $F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2}$. Some problems of combinatorics, we try to convert the problems into recursive form which is very helpful to solve the problems.

Example 1: If a_n represents the number of regions into which the plane is dived by n straight lines, such that no two lines are parallel and no three lines pass through same point.

Sol. We can easily observe that $a_1 = 2, a_2 = 4, a_3 = 7$.
Form above values, intuitively we think about a relation $a_n = a_{n-1} + n$
Let's try to find a proper reasoning for that.
Suppose there are $n - 1$ lines in a plane, when we draw nth line, it is cut by $n - 1$ lines into n segments and each segment divided the region into two parts, so 'n' segments create 'n' extra regions. Therefore $a_n = a_{n-1} + n$

Example 2: The figure shows a $1 \times n$ rectangle ABCD that is to be fully covered by 2 types of polygons of different sizes 1×3 and 1×1. What is the number of ways this can be done if $n = 4, 5, 6, 7$. Also, Find the recurrence relation for k_n, if k_n is the number of ways to cover $1 \times n$ rectangle.



Sol. Let k_n be the number of ways to cover $1 \times n$ rectangle with 1×3 and 1×1 polygons.
When $n = 4, k_4 = 1 + 2 = 3$
When $n = 5, k_5 = 1 + 3 = 4$

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When $n = 6, k_6 = 1 + 4 + 1 = 6$

When $n = 7, k_7 = 1 + 5 + 3 = 9$

We can see $k_7 = k_6 + k_4$

We can see that, covering of 1×7 achieved by either putting

Can we say that $k_n = k_{n-1} + k_{n-3}$?

In every arrangement of polygons, the last polygon is either 1×3 or 1×1.

Case I: If last polygon is 1×1

Number of arrangement of polygons in $1 \times n$ rectangle in which last polygon is 1×1 = Number of arrangements of polygons in $1 \times n - 1$ rectangle = k_{n-1}

Case II: If last polygon is 1×2

Number of arrangements in which last polygon is 1×3 = Number of arrangements of polygons, in $1 \times n - 3$ rectangle $= k_{n-3}$

$\Rightarrow k_n = k_{n-1} + k_{n-3}$

Example 3: Find the number of binary sequences of length 10 having no 11.

Sol. Let A_n denote the number of binary sequences of length n not containing 11 and M_n represents a sequence of length n not containing 11.

M_n can end with either 0 or 1.

If M_n end with 0, then it can be obtained by putting zero at the end of M_{n-1}.

If M_n end with 1, then $(n-1)^{th}$ digit must be 0, therefore it can be obtained by putting 01 at the end of M_{n-2}

$\Rightarrow A_n = A_{n-1} + A_{n-2}$, for $n > 2$

$A_1 = 2, A_2 = 3, A_3 = 5$

2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233

$A_{10} = 233$

Derangement

A derangement is an arrangement of the elements of a set, such that no element appears in its original position. For example, the derangements of ABC are BCA, CAB.

Number of derangements of n distinct items are $n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-.....+(-1)^n\frac{1}{n!}\right)$.

Proof: Let D(n) are the number of derangements of n distinct items. If item a_1 goes to position of a_r then the item a_r either goes to place of a_1 or any other place.

Case I: If a_r goes to a_1 then we have to derange the remaining $n - 2$ items = $D(n - 2)$

So the number of ways of doing this = $(n - 1)D(n - 1)$

Case II: If a_r goes to any other place then we have to derange the remaining $n - 1$ items = $D(n - 1)$

So the number of ways of doing this = $(n - 1)D(n - 2)$

$\Rightarrow D(n) = (n - 1)(D(n - 1) + D(n - 2))$

$D(n) - nD(n - 1) = -(D(n - 1) - (n - 1)D(n - 2)) = (-1)^2(D(n - 2) - (n - 2)D(n - 3)) = ... = (-1)^n(D(2) - 3D(1))$

$D(n) = nD(n - 1) + (-1)^n$

$$\frac{D(n)}{n!} = \frac{D(n-1)}{(n-1)!} + \frac{(-1)^n}{n!}$$

$$\sum_{n=2}^{n}\left(\frac{D(n)}{n!} - \frac{D(n-1)}{(n-1)!}\right) = \sum_{n=1}^{n}\frac{(-1)^n}{n!}$$

$$\frac{D(n)}{n!} - \frac{D(1)}{1!} = \frac{1}{2!} - \frac{1}{3!} + ... + (-1)^n\frac{1}{n!}$$

$$\frac{D(n)}{n!} - 0 = \frac{1}{2!} - \frac{1}{3!} + ... + (-1)^n\frac{1}{n!}$$

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$$D(n)=n!\left(\frac{1}{2!}-\frac{1}{3!}+\ldots+(-1)^n\frac{1}{n!}\right)$$

Example 4: If there are 4 letters and 4 corresponding envelopes. Find number of derangements such that no letter goes to the right envelope. How many ways are there such that only one letter goes to right envelope?

Sol. Derangement of 4 letters

$$=4!\left[1-\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}\right]=4!\left[\frac{1}{2}-\frac{1}{6}+\frac{1}{24}\right]$$

$$=24\left(\frac{12-4+1}{24}\right)=9$$

We require any one letter to go to its own envelope, such that remaining three will derange

Number of ways

$$={}^4C_1\times1\times\left[3!\left(1-\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}\right)\right]=4\times(3-1)=8$$

Example 5: Find the number of derangements of the integers from 1 to 10 inclusive, satisfying the condition that the set of elements in the first five places is

(a) 1, 2, 3, 4, 5 in some order.

(b) 6, 7, 8, 9, 10 in some order.

Sol. (a) The integers 1, 2, 3, 4, 5 can be put into the first five places in D_5 ways, because there are D_5 derangements of five things; the remaining integers from 6 to 10 can be put into the last five places in D_5 ways, so the answer is $D_5 \times D_5 = 1936$.

(b) Any arrangement of 6, 7, 8, 9, 10 in the first five places is a derangement, so there are 5! possibilities; the same is true for the integers 1, 2, 3, 4, 5 in the last five places. Hence the answer is $(5!)^2 = 14400$.

Exercise–7

1. There are 4 boxes numbered 1, 2, 3 and 4 and 4 balls numbered 1, 2, 3 and 4. Find the number of ways to put these balls into boxes(each box contain only one ball), so that
   (a) no ball goes to the box of the same number.
   (b) At least one ball goes to the box of same number.
2. In how many ways 6 letters can be placed in 6 envelopes such that
   (a) No letter is placed in its corresponding envelope.
   (b) At least 4 letters are placed in correct envelopes.
   (c) At most 3 letters are placed in wrong envelopes.
3. In how many ways five pairs of slippers can be distributed among five persons such that every person gets an odd pair?
4. How many numbers of ways are there such that a 2×10 chessboard can be perfectly covered by ten 1×2 rectangles without gaps and without overlaps?
5. Derive a Recurrence Relation with initial condition(RRwIC) for d_n, the number of sequence of the 26 upper case letters that do not contain DOG.
6. (AMC 2007) Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets of {1, 2, 3,..., 12}, including the empty set, are spacy ?
7. (AIME 2008) Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and AABAA, while BBAB is not such a sequence. How many such sequences have length 14?

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7.8 Pigeonhole Principle (PHP)

Principle 1: When m + 1 pigeons enter m pigeonholes (m is positive integer), there must be at least one hole having more than 1 pigeon.

Ex: In a group of 367 people, there must be two people with the same birthday. As there are 366 possible birthdays.

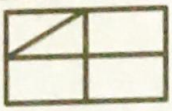
Example 1: In the morning, Ram draws socks randomly from his drawer. If there are 12 pairs of socks each pair of a different colour in the drawer, how many socks does Ram have to draw at most in order to get a matched pair?

Sol. There are 12 different colours (pigeonholes) and in order to ensure at least two socks (pigeons) have the same colour, number of socks drawn > 12.

So, at most 12 + 1 = 13 socks have to be drawn to get a matched pair.

Example 2: Five points are drawn randomly inside a unit square, show that there is at least two points with a distance less than $\frac{1}{\sqrt{2}}$.

Sol. Consider a unit square and cut it into 4 equal smaller squares, as shown below.



There are 4 equal smaller squares (pigeonholes) and 5 points (pigeons). By the pigeonhole principle, at least two points are in the same smaller square. The distance between these two points < length of the diagonal of the smaller square.

Length of the diagonal = $\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\frac{1}{\sqrt{2}}$

So, there are at least two points with a distance less than $\frac{1}{\sqrt{2}}$.

Principle 2: Let 'k' and 'n' be any two positive integers. If at least kn + 1 objects are distributed among 'n' boxes, then one of the boxes must contain at least k + 1 objects. In particular, if at least n + 1 objects are to be put into 'n' boxes, then one of the boxes must contain at least two objects.

Ex: A theater performs 7 plays in one season. Five women are there, each woman cast in 3 of the plays. Then some play has at least 3 women in its cast.

Number of pigeons $(5 \times 3) = 15$

Number of pigeonholes = 7

$k \times n + 1 = 2 \times 7 + 1$

3 or more pigeons in the same pigeonhole.

Ex: Given 8 different natural numbers, none greater than 14. Show that at least three pairs of them have the same difference

Try a set: 1, 2, 3, 7, 9, 11, 12, 13, 14.

Difference of 12 and 14 = 2.

Same for 9 and 11; 7 and 9; 1 and 3.

In this set, there are four pairs that all have the same difference.

Proof:

#pigeons (different pairs)= $C(8,2) = 8 \times 7/2 = 28$

#Pigeonholes (differences) = 14 – 1 = 13

Since $28 \geq k*n+1 = 2*13+1$, we have

3 or more pigeons in the same pigeonhole.

Principle 3: Let $q_1, q_2, ..., q_n$ be positive integers. If $q_1 + q_2 + ... + q_n - n + 1$ objects are distributed into 'n' boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 objects, ..., or the nth box contains at least q_n objects.

Proof: Suppose that we distribute $q_1 + q_2 + ... + q_n - n + 1$ objects among n boxes. If for each i = 1, 2, ... , n the i^{th} box contains fewer than q_i objects, then the total number of objects in all boxes does not exceed $(q_1 - 1) + (q_2 - 1) + ... + (q_n - 1)$ $= q_1 + q_2 + ... + q_n - n$.

Since this number is one less than the number of objects distributed, we conclude that for some i = 1, 2,..., n the i^{th} box contains at least q_i objects. Notice that it is possible to distribute $q_1 + q_2 + ... + q_n - n$ objects among n boxes in such a way that for no i = 1,2, ... , n is it true that the i^{th} box contains q_i or more objects. We do this by putting $q_1 - 1$ objects into the first box, $q_2 - 1$ objects into the second box, and so on.

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Example 3: In a computer science department, a student club can be formed with either 9 members from first year or 7 members from second year or 5 from third year or 3 from final year. What is the minimum number of students we have to choose randomly from department to ensure that a student club is formed?

Sol. $q_1 = 9, q_2 = 7, q_3 = 5, q_4 = 3$ and $n = 4$

Therefore, the minimum number of students required to ensure department club to be formed is $9 + 7 + 5 + 3 - 4 + 1 = 21$

Example 4: A box contains 6 red, 8 green, 10 blue, 12 yellow and 15 white balls. What is the minimum number of balls we have to choose randomly from the box to ensure that we get 9 balls of same color?

Sol. Here in this we cannot blindly apply pigeonhole principle. First, we will see what happens if we apply above formula directly. From the above formula we have get answer 47 because $6 + 8 + 10 + 12 + 15 - 5 + 1 = 47$ But it is not correct. In order to get the correct answer, we need to include only blue, yellow and white balls because red and green balls are less than 9. But we are picking randomly so we include it after we apply pigeonhole principle. i.e., 9 blue + 9 yellow + 9 white $- 3 + 1 = 25$ Since we are picking randomly so we can get all the red and green balls before the above 25 balls. Therefore, we add 6 red + 8 green + 25 = 39 We can conclude that in order to pick 9 balls of same color randomly, one has to pick 39 balls from a box.

Example 5: A box contains 6 red, 8 green, 10 blue, 12 yellow and 15 white balls. What is the minimum number of balls we

A team plays 12 games in a 10 days period and at least one game a day. We can always find a period of days in which exactly 8 games are played.

days	0	1	2	3	4	5	6	7	8	9	10
games	0	1	1	2	1	1	1	2	1	1	1
s_i	0	1	2	4	5	6	7	9	10	11	12
t_i	8	9	10	12	13	14	15	17	18	19	20

s_i = number of games played from day 1 to day i.

$0 \le s_i \le 12$, for $0 \le i \le 10$

$t_i = s_i + 8$, where $s_i \ne s_j, \forall\, i \ne j$; $t_i \ne t_j, \forall\, i \ne j$.

$8 \le t_i \le 20$, for $0 \le i \le 10$

Both the sequences s_i and t_i together take 22 values from 21 values(0 to 20). Therefore, by PHP at least two values are same. As neither two s_i are same not two t_i's are same. Therefore, $s_j = t_k = s_k + 8$ for some j and k.

$\Rightarrow$ 8 games are played from $(k + 1)^{th}$ day to j^{th} day, both days included.

Example 6: How many cards must be selected from a deck of 52 cards to make sure that at least 3 cards of the same suit are selected?

Sol. We have 4 suits (pigeonholes) and if all of them does not appear with at least 3 cards (pigeons), i.e. each suit has at most 2 cards, then the maximum number of cards that can possibly be drawn is (4).(2) = 8. So, the number of cards that must be drawn to ensure at least 3 cards are of the same suit is 8 + 1 = 9

Alternatively, let there be n cards drawn, then according to the generalized pigeonhole principle,

$$\left\lceil \frac{n}{4} \right\rceil = 3$$

$$\Rightarrow \frac{n}{4} > 3 - 1 = 2$$

$$\Rightarrow n > 1(4).(2) = 8.$$

So, at least 8 + 1 = 9 cards need to be drawn.

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Exercise–8

1. How many students do you need in a school to guarantee that there are at least 2 students who have the same first two initials?
2. Show that for any set of 10 points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance apart is at most $\sqrt{2}$.
3. A box contains three pairs of socks colored red, blue and white respectively. Suppose I take out the socks without looking at them. How many socks must I take out to be sure that they will include a matching pair?
4. Prove that for any given 50 positive integers, it is always possible to select out four numbers a_1, a_2, a_3 and a_4 from them, such that $(a_2 - a_1)(a_4 - a_3)$ is a multiple of 2009.
5. In a bag, there are some balls of the same size that are colored by 7 colors, and for each color the number of balls is 77. At least how many balls are needed to be picked out at random to ensure that one can obtain 7 groups of 7 balls each such that in each group the balls are homochromatic?
6. [SSSMO – 2001] A bag contain 200 marbles. There are 60 red ones, 60 blue ones, 60 green ones and the remaining 20 consist of yellow and white ones. If marbles are chosen from the bag without looking, what is the smallest number one must pick in order to ensure that, among the chosen marbles, at least 20 are of the same colour?
7. Prove that from any set of one hundred whole numbers, one can choose either one number which divisible by 100, or several numbers whose sum is divisible by 100.
8. (CHINA/2001) If 51 numbers are arbitrarily taken out from the first 100 natural numbers {1, 2, ..., 100}, prove that there must be two numbers drawn such that one is a multiple of the other.

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Things to Remember

Basic Counting Principles

(a) Fundamental Principle of Multiplication:

If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then the total number of ways in which both can occur in definite order are $m \times n$. This result can be extended to more than two events.

(b) Fundamental Principle of Addition:

If an event can occur in 'm' different ways and another event independent to the first can occur in 'n' different ways, then the total number of ways in which one of them can occur is $m + n$. This result can be extended to more than two events.

$n! = 1 \times 2 \times \ldots \times (n-1) \times n$

Note: $0! = 1$ (by definition), $n! = n((n-1)!)$

Selection of Distinct Objects

(i) Number of ways to select r different items from n different items are ${}^nC_r$.

Here, ${}^nC_r = \frac{n!}{r!(n-r)!}$. For example, ${}^{10}C_2 = \frac{10!}{2!8!}$

Note: ${}^nC_r = {}^nC_{n-r}$, ${}^nC_0 = 1$, ${}^nC_1 = n$

(ii) Total number of selections out of 'n' distinct objects includes the number of ways to select 0, 1, ..., n objects is equal to 2^n. Number of non-empty selections are $= 2^n - 1$.

Selection of Alike Items:

(a) Number of ways to select r identical items from n identical items = 1

(b) Total number of selections(selecting 0, 1, ..., n items) of n identical items = n + 1. This includes no selection also.

(c) Total number of selection of (a + b + c + d) items in which a are alike of one kind, b are alike of 2nd kind, c are alike of 3rd kind and d are distinct items $= (a+1)(b+1)(c+1)2^d$

Number of non-empty selections $= (a+1)(b+1)(c+1)2^d - 1$

Geometrical Counting Results:

(a) Number of points of intersection of 'n' non-concurrent and non-parallel lines is ${}^nC_2$.

(b) If there are 'n' points on a circle (or non collinear points), joining these points we get:

Number of lines = ${}^nC_2$

Number of Triangles = ${}^nC_3$

Number of r sided polygons = ${}^nC_r$

Number of diagonals of n sided polygon = ${}^nC_2 - n$

- Selection and Arrangement of r distinct items out of n distinct items = ${}^nC_r \times r!$.
- **Gap method:** Number of ways to arrange m + n distinct items in a row, so that no two of m things occur together = $n! \times {}^{n+1}C_m \times m!$
- **Block method:** Number of ways to arrange m + n things in a row, so that m things occur together = $m!(n+1)!$

Note: n(Number of arrangements when not all m things together) = n(Number of arrangements without restriction) – (Number of arrangements when all m are together)

- Number of ways of arranging 'p' elements in a row if out of 'p' elements 'm' are alike, 'n' are alike & rest are distinct, is given by $\frac{p!}{m! \times n!}$.

Divisors

Let $N = P_1^{n_1} \times P_1^{n_2} \times \ldots\ldots \times P_m^{n_m}$

(i) Number of Divisors $= (n_1+1)(n_2+1)\ldots\ldots(n_m+1)$

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(ii) Sum of Divisors $= \left(\frac{P_1^{n_1+1}-1}{P_1-1}\right)\times\left(\frac{P_2^{n_2+1}-1}{P_1-1}\right)\times.....\times\left(\frac{P_m^{n_m+1}-1}{P_m-1}\right)$

(iii) Number of ways in which N can be resolved as a product of two factors is

$$\begin{cases}\frac{1}{2}(n_1+1)(n_2+1).....(n_m+1); & \text{if N is not a perfect square}\\ \frac{1}{2}[(n_1+1)(n_2+1).....(n_m+1)+1]; & \text{if N is a perfect square}\end{cases}$$

(iv) Number of ways in which N can be resolved as a product of two coprime factors is $= 2^{m-1}$

(v) Sum of all the divisors of $a^m \times b^n \times c^q$ is

$(a^0 + a^1 + a^2 + ...+a^m) \times (b^0 + b^1 + b^2 ...+b^n) \times (c^0 + c^1 + c^2 + ...+c^q)$

Distribution of Distinct Objects

(a) Number of ways to distributing 'n' distinct objects among 'm' persons is m^n.

Remark: While distributing the distinct objects, always see the number of options that a given object to be distributed has.

(b) Number of ways of distributing m + n distinct items into two groups, one contains m items and other contains n items is $= \frac{(m+n)!}{m!n!}$, here $m \neq n$.

Note: If m = n, then number of ways $= \frac{(2n)!}{2!n!n!}$. Here, we divide the expression by 2!, because the groups are identical, so each distribution is counted 2! times.

Above result can be extended to any number of groups.

Note: If the groups further allotted to distinct persons or objects, then the number of ways of doing so = Number of ways to make groups × Arrangement of group among persons or objects.

(c) Number of ways of distributing 'n' identical objects among 'r' persons, if each person may receive any number of objects is equal to arranging n c's and r – 1 p's $= {}^{n+r-1}C_{r-1}$

Remark: This result is majorly applied in finding non-negative integral solutions of equations.

Number of non-negative integer solutions of $x_1 + x_2 + ...+ x_r = n$ is ${}^{n+r-1}C_{r-1}$.

Circular Permutation

The number of circular permutations of n distinct objects (n – 1)!.

Remark: If clockwise & anticlockwise arrangements are considered (as in necklace and Garland etc) than number of arrangements are $\frac{(n-1)!}{2}$.

Derangement

A derangement is an arrangement of the elements of a set, such that no element appears in its original position. For example, the derangements of ABC are BCA, CAB.

Number of derangements of n distinct items are $n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-.....+(-1)^n\frac{1}{n!}\right)$.

Pigeonhole Principle

Principle 1: When m + 1 pigeons enter m pigeonholes (m is positive integer), there must be at least one hole having more than 1 pigeon.

Principle 2: Let 'k' and 'n' be any two positive integers. If at least kn + 1 objects are distributed among 'n' boxes, then one of the boxes must contain at least k + 1 objects. In particular, if at least n + 1 objects are to be put into 'n' boxes, then one of the boxes must contain at least two objects.

Principle 3: Let $q_1, q_2, ..., q_n$ be positive integers. If $q_1 + q_2 + ...+q_n - n + 1$ objects are distributed into 'n' boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 objects, ..., or the nth box contains at least q_n objects.

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Practice Problems

1. [KVPY SX 2010] The number of rectangles that can be obtained by joining four of the twelve vertices of a 12-sided regular polygon is.
(a) 66 (b) 30
(c) 24 (d) 15

2. [KVPY SX 2010] How many six-digit numbers are there in which no digit is repeated, even digits appear at even places, odd digits appear at odd places and the number is divisible by 4?
(a) 3600 (b) 2700
(c) 2160 (d) 1440

3. [KVPY SX 2012] Three children, each accompanied by a guardian, seek admission in a school. The principal wants to interview all the 6 persons one after the other subject to the condition that no child is interviewed before its guardian. In how many ways can this be done.
(a) 60 (b) 90
(c) 120 (d) 80

4. [KVPY SX 2012] Let $S = \{1, 2, 3, ..., n\}$ and $A = \{(a, b) | 1 \le a, b \le n\} = S \times S$. A subset B of A is said to be a good subset if $(x, x) \in B$ for every $x \in S$. Then the number of good subsets of A is
(a) 1 (b) 2^n
(c) $2^{n(n-1)}$ (d) 2^{n^2}

5. [KVPY SX 2013] The number of integers n with $100 \le n \le 999$ and containing at most two distinct digits is
(a) 252 (b) 280
(c) 324 (d) 360

6. [KVPY SX 2013] In a tournament with five teams, each team plays against every other team exactly once. Each game is won by one of the playing teams and the winning team scores one point, while the losing team scores zero. Which of the following is NOT necessarily true?
(a) There are at least two teams which have at most two points each.
(b) There are at least two teams which have at least two points each.
(c) There are at most three teams which have at least three points each
(d) There are at most four teams which have at most two points each

7. [KVPY SX 2014] For each positive integer n, let $A_n = \max\left\{\binom{n}{r} \middle| 0 \le r \le n\right\}$. Then the number of elements n in {1, 2, ..., 20} for which $1.9 \le \frac{A_n}{A_{n-1}} \le 2$ is
(a) 9 (b) 10
(c) 11 (d) 12

8. [KVPY SX 2014] Let $S = \{(a, b): a, b \in Z, 0 \le a, b \le 18\}$. The number of elements (x, y) in S such that $3x + 4y + 5$ is divisible by 19 is
(a) 38 (b) 19
(c) 18 (d) 1

9. [KVPY SX 2014] Let X be a set of 5 elements. The number d of ordered pairs (A, B) of subsets of X such that $A \ne \varphi$, $B \ne \varphi$, $A \cap B = \varphi$ satisfies
(a) $50 \le d \le 100$ (b) $101 \le d \le 150$
(c) $151 \le d \le 200$ (d) $201 \le d$

10. [KVPY SX 2016] The number of noncongruent integer-sided triangles whose sides belong to the set {10, 11, 12, ..., 22} is-
(a) 283 (b) 446
(c) 448 (d) 449

11. [KVPY SX 2016] Three schools send 2, 4 and 6 students, respectively, to a summer camp. The 12 students must be accommodated in 6 rooms numbered 1, 2, 3, 4, 5, 6 in such a way that each room has exactly 2 students and both from the same school. The number of ways, the students can be accommodated in the rooms is-
(a) 60 (b) 45
(c) 32400 (d) 2700

12. [KVPY SX 2017] The number of 5-tuples (a, b, c, d, e) of positive integers such that
I. a, b, c, d, e are the measures of angles of a convex pentagon in degrees
II. $a \le b \le c \le d \le e$
III. a, b, c, d, e are in arithmetic progression is
(a) 35 (b) 36
(c) 37 (d) 126

13. [KVPY SX 2017] Let A be the set of all permutations $a_1, a_2, ..., a_6$ of 1, 2, ..., 6 such that $a_1, a_2, ..., a_k$ is not a permutation of 1, 2, ..., k for any k, $1 \le k \le 5$. Then the number of elements in A is
(a) 192 (b) 408
(c) 312 (d) 528

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14. [KVPY SA 2017] Consider the set A = {1, 2, 3, ..., 30}. The number of ways in which one can choose three distinct numbers from A so that the product of the chosen numbers is divisible by 9 is [2017]
(a) 1590 (b) 1505
(c) 1110 (d) 1025

15. [KVPY SA 2018] Let m (respectively, n) be the number of 5-digit integers obtained by using the digits 1, 2, 3, 4, 5 with repetitions (respectively, without repetitions) such that the sum of any two adjacent digits is odd. Then m/n is equal to
(a) 9 (b) 12
(c) 15 (d) 18

16. [KVPY SA 2019] The number of non-negative integer solutions of the equations $6x + 4y + z = 200$ and $x + y + z = 100$ is
(a) 3 (b) 5
(c) 7 (d) Infinite

17. [KVPY SX 2019] The number of four-letter words that can be formed with letters a, b, c such that all three letters occur is
(a) 30 (b) 36
(c) 81 (d) 256

18. [KVPY SX 2019] Let E denote the set of letters of the English alphabet, V = {a, e, i, o, u} and C be the complement of V in E. Then, the number of four-letter words (where repetitions of letters are allowed) having at least one letter from V and at least one letter from C is
(a) 261870 (b) 314160
(c) 425880 (d) 851760

19. Find the number of times the digit 5 will be written when listing integers from 1 to 1000.

20. A seven digit number divisible by 9 is to be formed by using 7 out of numbers {1, 2, 3, 4, 5, 6, 7, 8, 9}. Find the number of ways in which this can be done.

21. (a) A 5 digit number divisible by 3 is to be formed using the numerals 0, 1, 2, 3, 4 & 5 without repetition. Find the total number of ways.
(b) How many five digits numbers divisible by 3 can be formed using the digits 0, 1, 2, 3, 4, 7 and 8 if each digit is to be used at most once?

22. A committee of 5 is to be chosen from a group of 9 people. Find the number of ways in which it can be formed if two particular persons either serve together or not at all and two other particular persons refuse to serve with each other.

23. In a certain algebra exercise book there are 4 examples on arithmetical progressions, 5 examples of permutation - combination and 6 examples on binomial theorem. Find the number of ways a teacher can select for his pupils atleast one but not more than 2 examples from each of these sets.

24. How many ordered pairs (P, Q) of subsets of {a, b, c, d, e} can be made such that:
(a) $P \cap Q = \phi$
(b) $P \cap Q = \{a\}$
(c) $P \cap Q$ Contains exactly 2 elements

25. A seven digit number with distinct digits is in form of abcdefg (g, f, e, etc. are digits at units, tens, hundred place etc.) where $a < b < c < d > e > f > g$. Find the number of such numbers.

26. The number of ways of arranging 'm' number out of 1, 2, 3, ..., n so that maximum is (n - 2) and minimum is 2 (repetitions of numbers is allowed) such that maximum and minimum both occur exactly once, (n > 5, m > 3) is
(a) ${}^{n-3}C_{m-2}$
(b) ${}^{m}C_{2}\,(n-3)^{m-2}$
(c) $m\,(m-1)\,(n-5)^{m-2}$
(d) ${}^{n}C_{2} \cdot {}^{n}C_{m}$

27. Total number of ways of selecting two numbers from the set {1, 2, 3, 4, ..., 3n} so that their sum is divisible by 3 is equal to:

28. There are m points on a straight line AB and n points on the line AC none of them being the point A. Triangles are formed with these points as vertices, when
(i) A is excluded (ii) A is included
The ratio of number of triangle in the two cases is:

29. In a plane there are 6 lines, no two of which are parallel & no three are concurrent. How many triangles can be formed with their points of intersection as vertices.

30. The number of ways in which we can choose 3 squares of unit area on a chess board such that one of the squares has its two sides common to other two squares.

31. Given 11 points, of which 5 lies on one circle, other than these 5, no 4 lie on one circle. Then the maximum number of circles that can be drawn so that each contains atleast three of the given points is _____.

32. Two classrooms A and B having capacity of 25 and (n - 25) seats respectively. A_n denotes the number of possible seating arrangements of room 'A', when 'n' students are to be seated in these rooms starting from room 'A' which is to be filled up full to its capacity. If $A_n - A_{n-1} = 25!\,({}^{49}C_{25})$ then 'n' equals _____.

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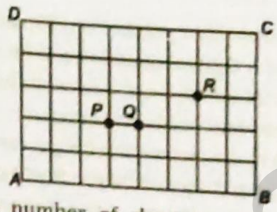
33. In how many ways it is possible to select six letters, including at least one vowel from the letters of the word "F L A B E L L I F O R M". (It is a picnic spot in U. S. A.)
34. The total number of permutations of 4 letters that can be made out of the letters of the word EXAMINATION is _____.
35. Number of ways in which letters of the word ENGINEER can be arranged:
    (a) so that the word starts with E but does not end with N.
    (b) so that the word neither starts with E nor ends with N.
    (c) so that vowels occur in alphabetical order.
    (d) so that no two alike letters are together.
36. The number of divisors of $2^3 . 3^3 . 5^3 . 7^5$ of the form $4n + 1, n \in N$ is _____.
37. A disarranged number from the set 1 – 9 is an arrangement of all these number so that all numbers take up its unusual position. (e.g. 1 is in any place other than the first position, 2 is in any place other than the second position ..... all the way to 9). Number of ways in which at least six number stake up their usual positions, is _____.
38. How many ten digits whole number satisfy the following property they have 2 and 5 as digits, and there are no consecutive 2's in the number (i.e. any two 2's are separated by at least one 5)?
39. How many 4 digit numbers are there which contains not more than 2 different digits?
40. There are counters available in 7 different colours. Counters are all alike except for the colour and they are at least ten of each colour. Find the number of ways in which an arrangement of 10 counters can be made. How many of these will have counters of each colour.
41. The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1,2 and 3 only, is _____.
42. Let the product of all the divisors of 1440 be P. If P is divisible by 24^x, then the maximum value of x is _____.
43. The number of three-digit numbers of the form xyz such that $x < y$ and $z \leq y$ is _____.
44. Consider the network of equally spaced parallel lines(6 horizontal and 9 vertical) shown in the figure. All small squares are of the same size. A shortest route from A to C is defines as a route consisting of 8 horizontal steps and 5 vertical steps. Find



    (a) The number of shortest routes through the junction P.
    (b) The number of shortest routes which go following street PQ.
    (c) The number of shortest routes which passes through junction P and R.
45. The minimum marks required for clearing a certain screening paper is 210 out of 300. The screening paper consists of '3' sections each of Physics, Chemistry, and Maths. Each section has 100 as maximum marks. Assuming there is no negative marking and marks obtained in each section are integers, find the number of ways in which a student can qualify the examination is (Assuming no cut-off limit).
46. 10 identical balls are to be distributed in 5 different boxes kept in a row and labelled A, B, C, D and E. Find the number of ways in which the balls can be distributed in the boxes if no two adjacent boxes remain empty.
47. Find how many three digit numbers, lying between 100 and 999 inclusive, have two and only two cosecutive digits identical.
48. Let X = {1, 2, 3,..., 17}. Find the number of subsets Y of X with odd cardinalities.
49. How many integers are there between 0 and 10^5 having the digit sum of equal to 8?
50. Find the largest positive integer n such that n! ends with exactly 100 zeroes.
51. Suppose 40 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?
52. Five students A, B, C, D and E from a team to take part in a 5 leg relay competition. If A cannot run the first leg and D cannot run the last leg, how many ways can we arrange them to run the relay?
53. Suppose that each of n people knows exactly one piece of information, and all n pieces are different. Every time person A phones person B, A tells B everything he knows, while B tells A nothing. What is the minimum of phones called between pairs of people needed for everyone to know everything?

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54. For each positive integer n, let a_n denote the number of n digit integers formed by some or all of the digits 0,1,2 and 3 which contain neither a block of 12 nor a block of 21. Evaluate a_9.

55. As shown in the picture, the knight can move to any of the indicated squares of the 8 × 8 chessboard in 1 move. If the knight starts from the position shown, find the number of possible landing positions after 20 consecutive moves.

56. How many four digit numbers greater than 5000 can be formed from the digit 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if only the digit 4 may be repeated?

57. There girls A,B and C, and nine boys are to be lined up in a row. Let n be the number of ways ths can be done if B must lie between A and C, and A, B must be separated by exactly 4 boys. Determine [n/7!]

58. Determine the number of positive integer divisors of 998^{4999} that are not the divisors of 998^{4998}

59. Consider a 2008 x 2008 chess board. Let M be the smallest number of rectangles that can be drawn on the chess board so that the sides of every cell of the board is contained in the sides of the one of the rectangles. Find the value of M. (For example, for the 2 × 3 chess board, the value of M is 3).

60. Determine the number of 8-digit positive integers such that after deleting any one digit, the remaining 7-digit number is divisible by 7.

61. In how many ways can a team of 6 horses be selected out of a stud of 16, so that there shall always be 3 out of A B C A' B'C', but never AA', BB' or CC' together.

62. There are n straight lines in a plane, no 2 of which parallel, & no 3 pass through the same point. Their point of intersections are joined. Show that the number of fresh lines thus introduced is n(n -1)(n - 2)(n - 3)/8

63. Prove that if each of m points in one straight line be joined to each of n in another by straight lines terminated by the points, then excluding the given points, the lines will intersect ¼ mn(m – 1)(n –1) times.

64. Find the number of words each consisting of 3 consonants & 3 vowels that can be formed from the letters of the word "circumference". In how many of these all 'c's will be together.

65. In how many different ways can 15 Candy bars be distributed between Ram, Shyam, Ghanshyam and Balram, if Ram can not have more than 5 candy bars and Shyam must have at least two. Assume all Candy bars to be alike.

66. How many integers between 1000 and 9999 have exactly one pair of equal digit such as 4049 or 9902 but not 4449 or 4040?

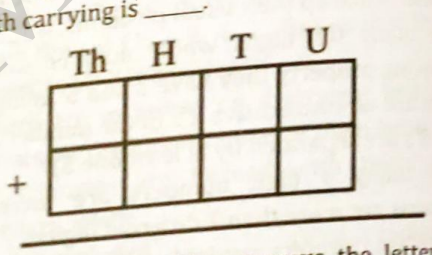
67. In a box there are 10 balls, 4 red, 3 black, 2 white and 1 yellow. In how many ways can a child select 4 balls out of these 10 balls? (Assume that the balls of the same color are identical)

68. In a certain test, a_i students gave wrong answers to at least i question, where i = 1, 2,...., k. No student gave more that k wrong answers. Find the total number of wrong answers given.

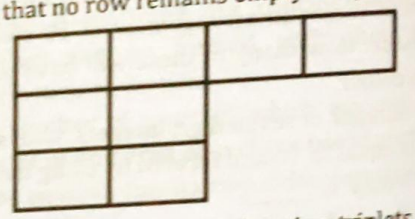
69. Use permutation or otherwise, prove that $\frac{n^2!}{(n!)^n}$ is an integer, where n is a positive integer.

70. Number of positive integral solutions satisfying the equation $(x_1 + x_2 + x_3)(y_1 + y_2) = 77$, is _____.

71. In the figure, two 4-digit numbers are to be formed by filling the places with digits. The number of different ways in which the places can be filled by digits so that the sum of the numbers formed is also a 4-digit number and in no place the addition is with carrying is _____.



72. The number of different ways the letters of the word VECTOR can be placed in 8 boxes given below such that no row remains empty is equal to _____.



73. The number of ways of choosing triplets (x, y, z) such that $z \geq \max\{x, y\}$ and $x, y, z \in \{1, 2, \ldots, n\}$ is _____.

74. Total number of positive integral solution of $15 < x_1 + x_2 + x_3 \leq 20$ is equal to _____.

75. The number of ways in which we can choose 2 distinct integers from 1 to 100 such that difference between them at most 10 is:

(a) $^{40}C_2$ (b) $^{70}C_2$

(c) $^{100}C_2 - {}^{99}C_2$ (d) None of these

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76. Number of different words that can be formed using all the letters of the word "DEEPMALA" if two vowels are together and the other two are also together but separated from the first two is:
(a) 960 (b) 1200
(c) 2160 (d) 1440

77. The number of permutations of the letters of the word HINDUSTAN such that neither the pattern 'HIN' nor 'DUS' nor 'TAN' appears, are _____.

78. There are 720 permutation of the digits 1, 2, 3, 4, 5, 6. Suppose these permutations are arranged from smallest to largest numerical values, beginning from 1 2 3 4 5 6 and ending with 6 5 4 3 2 1.
(a) What number falls on the 124th positions?
(b) What is the position of the number 321546?

79. A forecast is to be made of the results of five cricket matches, each of can be win, a draw or a lost for Indian team. Find
(i) the number of different possible forecasts
(ii) the number of forecasts containing 0, 1, 2, 3, 4 and 5 errors respectively

80. In Indo-Pak one day International cricket match at Sharjah, India needs 14 runs to win just before the start of the final over. Find the number of ways in which India just manages to win the match (i.e. scores exactly 14 runs), assuming that all the runs are made off the bat & the batsman cannot score more than 4 runs off any ball.

81. A question paper on mathematics consists of twelve questions divided into three parts. A, B and C, each containing four questions. In how many ways can an examinee answer five questions, selecting at least one from each part.

82. The number of ways in which 8 distinguishable apples can be distributed among 3 boys such that every boy should get at least 1 apple and at most 4 apples is K. ${}^{7}P_{3}$ where K has the value equal to _____.

83. There are 12 books on Algebra and Calculus in our library, the books of the same subject being different. If the number of selections each of which consists of 3 books on each topic is greatest, then the number of books of Algebra and Calculus in the library are respectively _____.

84. A gentleman invites a party of $m + n$ ($m \neq n$) friends to a dinner and places m at one table T_1 and n at another table T_2, the table being round. If not all people shall have the same neighbour in any two arrangement, then the number of ways in which he can arrange the guests, is _____.

85. Find the number of integers between 1 and 10000 with at least one 8 and at least one 9 as digits.

86. Sameer has to make a telephone call to his friend Harish, Unfortunately, he does not remember the 7 digit phone number. But he remembers that the first three digits are 635 or 674, the number is odd and there is exactly one 9 in the number. The maximum number of trials that Sameer has to make to be successful is _____.

87. Six faces of an ordinary cubical die marked with alphabets A, B, C, D, E and F is thrown n times and the list of n alphabets showing up are noted. Find the total number of ways in which among the alphabets A, B, C, D, E and F only three of them appear in the list.

88. Find the number of integers in the set {1, 2, 3,..., 2009} whose sum of the digits is 11.

89. A four digit number consists of two distinct pairs of repeated digits (for example 2211, 2626 and 7007). Find the total number of such possible numbers that are divisible by 7 or 101 but not both.

90. Using the digits 0,1,2,3 and 4, find the number of 13 digit sequence that can be written so that the difference between any two consecutive digits is 1. Examples of such 13 – digit sequence are 0123432123432, 2323432321234 and 23010101234323

91. Three are eight envelopes numbers 1 to 8. Find the number of ways in which 4 identical red buttons and 4 identical blue buttons can be put in the envelops such that each envelope contains exactly one button, and the sum of the numbers on the envelops the red buttons is more than the sum of the numbers on the envelopes containing the blue buttons.

92. Find the number of positive divisors of $(2008^3 + (3 \times 2008 \times 2009) + 1)^2$

93. Find the number of 2-element subsets {a, b} of {1,2,3,...,99, 100} such that ab + a + b is a multiple of 7.

94. Find the number of 0–1 binary sequences for by six 0's and six 1's such that no three 0's are together. For example,110010100101 is such a sequence but 101011000101 and 11010110001 are not.

95. In each of the following 6-digit positive integers 555555,555333,818811, 300388 every digit in the number appears at least twice. Find the number of such 6-digit positive integers.

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96. Using the digits 1,2,3,4,5,6,7,8, we can form 8!(=40320) 8-digit numbers in which the eight digits are all distinct. For $1 \le k \le 40320$, let "a" denote the kth number if these numbers are arranged in increasing order:
12345678, 12345687, 12345768,.....87654321 ;
that is $a_1 = 12345678$, $a_2 = 1245687$, $....a_{40320} = 87654321$. Find $a_{2009} - a_{2008}$

97. Let A be an n -element subset of {1, 2,.....,2009} with the property that the difference between any two numbers in A is not a prime number. Find the largest possible value of n. Find a set with this number of elements. (Note: 1 is not a prime number)

98. Let S = {1, 2, 3,....., 30}. Determine the number of vectors (x, y, z, w) with x, y, z, w ∈ S such that $x < w$ and $y < z < w$.

99. Let f(n) be the number of 0's in the decimal representation of the positive integer n. For example f(10001123) = 3 and f (1234567) = 0. Find the value of f(1) + f(2) + f(3) +....f(99999)

100. Let S= {1.2,3,4,...16}. In each of the following subsets of S.
{6}, {1,2,3}, {5,7,9,10,11,12}, {1,2,3,4,5,6,7,8,9}
The sum of the all the elements is a multiple of 3. Find the total number of non empty subsets A of S such that the sum of all elements in A is a multiple of 3.

101. Determine the number of ways of tilling a 4 x 9 rectangle by tiles of size 1×2.

102. Find the number of 7 digit positive integer such that the digits from left to right non increasing (Examples of 7 digit non increasing numbers are 9998766 and 555555; An example of a number that is NOT non increasing is 7776556).

103. Determine the number of 4-elements subsets {a, b, c, d} of {1, 2, 3, 4,..., 20} such that a + b + c + d is divisible by 3.

104. There are 5 white, 4 yellow, 3 green, 2 blue & 1 red balls. The balls are all identical except for color. Five of these are to be arranged in a line. Find the number of distinct arrangements.

105. There are 20 books on Algebra & Calculus in our library. Prove that the greatest number of selections each of which consists of 5 books on each topic is possible only when there are 10 books on each topic in the library.

106. 6 white & 6 black balls of the same size are distributed among 10 different urns. Balls are alike except for the color & each urn can hold any number of balls. Find the number of different distributions of the balls so that there is at least 1 ball in each urn.

107. How many 15 letter arrangements of 5 A's, 5 B's and 5 C's have no A's in the first 5 letters, no B's in the next 5 letters, and no C's in the last 5 letters?

108. Let a_n denote the number of all n-digit positive integers formed by the digits 0, 1 or both such that no consecutive digits in them are 0. Let b_n = The number of such n-digit integers ending with digit 1 and c_n = The number of such n-digit integers ending with digit 0.
i. Which of the following is correct?
(a) $a_{17} = a_{16} + a_{15}$ (b) $c_{17} \ne c_{16} + c_{15}$
(c) $b_{17} \ne b_{16} + c_{16}$ (d) $a_{17} = c_{17} + b_{16}$
ii. The value of b_6 is
(a) 7 (b) 8
(c) 9 (d) 11

109. In an examination the maximum marks for each of three papers is n and that for fourth paper is 2n. If marks obtained in each paper are whole numbers, then the number of ways in which a candidate can get 3n marks is _____.

110. Determine the number of 3-digit numbers in base 10 having
(i) At least one 5 and having no 3 and
(ii) At least one 5 and having exactly one 3 separately.

111. Find the number of combinations n together of 3n letters of which n are 'a' and n are 'b' and the rest unlike.

112. Let A be any k element subset of the set {1, 2, 3, 4,... 100}. Determine the minimum value of k such that we can always guarantee the existence of two numbers a and b in A such that $|a-b| \le 4$

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Pre RMO PYQ's

1. [PRMO 2020 KV] Find the number of positive integers n such that the highest power of 7 dividing n! is 8.
2. [PRMO 2020 KV] Ria has 4 green marbles and 8 red marbles. She arranges them in a circle randomly, if the probability that no two green marbles are adjacent is $\frac{p}{q}$ where the positive integers p & q have no common factors other than 1, what is p + q?
3. [PRMO 2020 KV] Let A = {1, 2, 3, 4, 5, 6, 7, 8,}, B = {9, 10, 11, 12, 13, 14, 15, 16} and C = {17, 18, 19, 20, 21, 22, 23, 24}. Find the number of triples (x, y, z) such that $x \in A$, $y \in B$. $z \in C$ and $x + y + z = 36$.
4. [PRMO 2020 KV] Consider a permutation $(a_1, a_2, a_3, a_4, a_5)$ of {1,2,3,4,5}. We say the 5-tuple $(a_1, a_2, a_3, a_4, a_5)$ is flawless if for all $1 \leq i < j < k \leq 5$, the sequence (a_i, a_j, a_k) is not an arithmetic progression (in that order). Find the number of flawless 5-tuples.
5. [PRMO 2020 KV] Ari chooses 7 balls at random from n balls numbered 1 to n. If the probability that no two of the drawn balls have consecutive numbers equals the probability of exactly one pair of consecutive numbers in the chosen balls, find n.
6. [PRMO 2020] Three couples sit for a photograph in 2 rows of three people each such that no couple is sitting in the same row next to each other or in the same column one behind the other. How many arrangements are possible?
7. [PRMO 2020] How many two-digit numbers have exactly 4 positive factors? (Here 1 and the number n are also considered as factors of n.)
8. [PRMO 2020] In the figure below 4 of the 6 disks are to be colored black and 2 to be colored white. Two colorings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same.

There are only four such colorings for the given two colors, as shown in Figure 1. In how many ways can we colour the 6 disks such that 2 are colored black, 2 are colored white, 2 are colored blue with the given identification condition?

9. [PRMO 2020] A bug travels in the coordinate plane moving only along the lines that are parallel to the x axis and y axis. Let A = (−3, 2) and B(3, −2). Consider all possible paths of the bug from A to B of length at most 14. How many points with integer coordinates lie on at least one of these paths?
10. [PRMO 2020] A natural number n is said to be good if n is the sum of r consecutive positive integers, for some $r \geq 2$. Find the number of good numbers in the set {1, 2, ..., 100}.
11. [PRMO 2019] Find the number of positive integers less than 101 that can not be written as the difference of two square of integers.
12. [PRMO 2019] Let N be the number of ways of choosing a subset of 5 distinct numbers from the set $\{10a + b: 1 \leq a \leq 5, 1 \leq b \leq 5\}$ Where a, b are integers, such that no two of the selected numbers have the same units digit and no two have the same tens digits. What is the remainder when N is divided by 73?
13. [PRMO 2019] What is the smallest prime number p such that $p^3 + 4p^2 + 4p$ has exactly 30 positive divisors?
14. [PRMO 2019] Five persons wearing badges with number 1, 2, 3, 4, 5 are seated on 5 chairs
15. around a circular table. In how many ways can they be seated so that no two persons whose badges have consecutive numbers are seated next to each other? (Two arrangements obtained by rotation around the table are considered different.)
16. [PRMO 2019] In how many ways can a pair of parallel diagonals of a regular polygon of 10 sides be selected?
17. [PRMO 2019] A $1 \times n$ rectangle ($n \geq 1$) is divided into n unit (1×1) squares. Each square of this rectangle is coloured red, blue or green. Let f(n) be the number of colourings of the rectangle in which there are an even number of red squares. What is the largest prime factor of f(9)/f(3)? (The number of red squares can be zero)
18. [PRMO 2019] We will say that a rearrangement of the letters of a word has no fixed letters if. When the rearrangement is placed directly below the word, no column has the same letter repeated. For instance, H B R A T A is a rearrangement with no fixed letters of B H A R A T have? (The two A's are considered identical)

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19. [PRMO 2019] Let E denote the set of all natural numbers n such that $3 < n < 100$ and the set {1, 2, 3, ..., n) can be partitioned in to 3 subsets with equal sums. Find the number of elements of E.
20. [PRMO 2019] There are several tea cups in the kitchen, some handle and the others without handles. The number of ways of selecting two cups without a handle and three with a handle is exactly 1200. What is the maximum possible number of cups in the kitchen?
21. [PRMO 2018] Determine the number of 8-tuples $(\epsilon_1, \epsilon_2, ..., \epsilon_8)$ such that $\epsilon_1, \epsilon_2, ..., \epsilon_8 \in \{1, -1\}$ and $\epsilon_1 + 2\epsilon_2 + 3\epsilon_3 + ... + 8\epsilon_8$ is a multiple of 3.
22. [PRMO 2018] A positive integer k is said be good if there exists a partition of {1, 2, 3,....., 20} in to disjoint proper subsets such that the sum of the numbers in each subset of the partition is k. How many good numbers are there?
23. [PRMO 2018] If N is the number of triangles of different shapes (i.e. not similar) whose angles are all integers (in degrees), what is N/100?
24. [PRMO 2018] What is the number of ways in which one can choose 60 units square from a 11×11 chessboard such that no two chosen square have a side in common?
25. [PRMO 2018] What is the number of ways in which one can colour the square of a4 $\times$ 4 chessboard with colours red and blue such that each row as well as each column has exactly two red squares and blue squares?
26. [PRMO 2018] Let N be the number of ways of distributing 8 chocolates of different brands among 3 children such that each child gets at least one chocolate, and no two children get the same number of chocolates. Find the sum of the digits of N.
27. [PRMO 2017] There are five cities A, B, C, D, E on a certain island. Each city is connected to every other city by road. In how many ways can a person staring from city A come back to A after visiting some cities without visiting a city more than once and without taking the same road more than once? (The order in which he visits the cities also matters e.g., the routes $A \to B \to C \to A$ and $A \to C \to B \to A$ are different.)
28. [PRMO 2017] There are eight rooms on the floor of a hotel, with four rooms on each side of corridor, symmetrically situated (that is each ro is exactly opposite to one other room). Four gue have to be accommodated in four of the eig rooms (that is, one in each) such that no two gue are in adjacent rooms or in opposite rooms. In how many ways can the guests be accommodated?
29. [PRMO 2017] Find the number of ordered trip (a, b, c) of positive integers such that $abc = 180$.
30. [PRMO 2017] Suppose in the plane 10 pair nonparallel lines intersect one another. What is the maximum possible number of polygons (with finite areas) that can be formed?
31. [PRMO 2015] How many segments have both the endpoints located at the vertices of a given cube?
32. [PRMO 2015] Let E(n) denote the sum of even digits of n. For example, $E(1243) = 2 + 4 = 6$. What is the value of $E(1) + E(2) + E(3) + ... + E(100)$?
33. [PRMO 2015] At a party, each man danced with exactly four women and each woman danced with exactly three men. Nine men attended the party. How many women attended the party?
34. [PRMO 2015] A subset B of set of first 100 positive integers has the property that no two elements of B sum to 125. What is the maximum possible number of elements in B?
35. [PRMO 2014] What is the number of ordered pairs (A, B) where A and B are subsets of {1, 2, ..., 5} such that neither $A \subseteq B$ nor $B \subseteq A$?
36. [PRMO 2013] There are n – 1 red balls, n green balls and n + 1 blue balls in a bag. The number of ways of choosing two balls from the bag that have different colours is 299. What is the value of n?
37. [PRMO 2013] To each element of the set S = {1, 2, ..., 1000} a colour is assigned. Suppose that for any two elements a, b of S, if 15 divides a + b then they are both assigned the same colour. What is the maximum possible number of distinct colours used?
38. [PRMO 2012] A postman has to deliver five letters to five different houses. Mischievously, he posts one letter through each door without looking to see if it is the correct address. In how many different ways could he do this so that exactly two of the five houses receive the correct letters?

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